Revisiting brittle fracture as an energy minimization problem : Comparisons of Griffith and Barenblatt surface energy models

M. Charlotte*, G. Francfort*, J.-J. Marigo* and L. Truskinovsky**

* Laboratoire des Propriétés Mécaniques et Thermodynamiques des Matériaux (UPR-CNRS 9001)
Institut Galilée, Université Paris-Nord, Avenue J.B. Clément, 93430 Villetaneuse

** Department of Aerospace Engineering and Mechanics, University of Minnesota
110, Union Street, 107, Akerman Hall, Minneapolis, MN 55455, USA

ABSTRACT :

Following the ideas presented in Francfort and Marigo [1], we assume that the cracking of a brittle body is governed by a principle of least energy. By adopting Griffith's assumption on the surface energy and by seeking for global minimum, we showed that it is then possible to remedy to some major defects of the classical Griffith theory based on the criterion of the critical energy release rate, in particular the issue of the crack initiation. We can in this way completely solve various problems of failure of engineering structures, either by using analytical methods (see [1], [2] and [3]) or by using an adapted numerical method (see [4] or [5]). In counterpart this new formulation leads itself to a few but not admissible defects like undesirable size effects or the inability of the material to sustain body forces. As a remedy, we simply propose to change the form of the surface energy by adopting the idea of Barenblatt and to seek local minima. This paper is devoted to analyze the advantages of these changes in a one dimensional context. It completes the previous published works of [6], [7] or [8] about Barenblatt model.

1 Introduction

We consider a homogeneous bar of natural length L, with cross-sectional area S, constituted of an elastic breakable material with Young modulus E and the breakable properties of which will be given below. Before any loading, the bar is assumed to be sound.

1.1 The loading cases.

We will consider three types of loading for the bar :

1. **Prescribed displacements.** The end $x = 0$ is fixed, while the displacement of the end $x = L$ is prescribed to a value $\pm U$, $+U$ in the case of a tensile test and $-U$ in the case of a compressive test, in any case the magnitude U increasing from 0.

2. **Prescribed surface traction.** The end $x = 0$ is fixed, while the end $x = L$ is submitted to either a tensile or a compressive force $\pm F$, $+F$ in traction and $-F$ in compression, the magnitude F increasing from 0.

3. **Prescribed body forces.** The end $x = 0$ is fixed, the end $x = L$ is free and the bar is submitted to an uniform distribution of body forces $\pm g$, $+g$ in tension or $-g$ in compression, the magnitude g of which is increasing from 0.

1.2 The set of admissible displacements.

To take into account the fracture of the bar, we must consider a class of displacement fields larger than that used in classical elasticity. In particular, when the bar breaks at a point x, then the displacement u may suffer a jump discontinuity $[\![u]\!](x)$ at this point. Moreover since this point is not known in advance, we must envisage displacement fields with jump discontinuities anywhere in the (closed) interval $[0, L]$[1].

[1] The bar can break at one end, see for example the case of prescribed body forces with Barenblatt's model.

From a mathematical point of view, the most convenient space of displacement fields in the present context of fracture with free points of discontinuity is the space $BV(0,L)$ of function of bounded variation, see for example [9] for an introduction to this space and [10] for an application in 1D fracture mechanics. However, in the present paper, for the sake of simplicity of the presentation we will limit our attention to piecewise smooth displacement fields, that is to functions u continuous and differentiable anywhere but at a finite number of points of $[0,L]$, their set is denoted by $S(u)$, where u is discontinuous. The classical derivative of u, defined in $[0,L]\setminus S(u)$, will be denoted by u'. Because of the possibility of a jump at 0 or L and in order to give a precise definition of the boundary conditions, we must extend the domain of definition of the fields to the whole real line. At this effect, we put

$$u(x) = u^-(0) = 0 \quad for \quad x < 0, \tag{1}$$

$$u(x) = u^+(L) \quad for \quad x > L, \tag{2}$$

where $u^+(L)$ is equal to the prescribed displacement U at the end in the first case of loading or is an arbitrary constant in the other ones. In order to preserve the orientation of the matter and to prevent the interpenetration of the lips of the crack, we must prohibit negative jumps and for that we introduce the following inequality that any admissible displacement field u must satisfy :

$$[\![u]\!](x) \geq 0, \quad \forall x \in S(u). \tag{3}$$

In the definition of a local minimum, the notion of neighborhood of a displacement u appears. That needs to equip this space of piecewise smooth functions with a norm. Here we choose the following one

$$\|u\| = \int_{-\infty}^{+\infty} |u'(x)|dx + \sum_{S(u)} |[\![u]\!](x)|. \tag{4}$$

1.3 The surface energy.

1. **Griffith's model :** If we adopt the idea of Griffith, see [11], then the surface energy associated to a crack, that is a surface across which the displacement jumps, is proportional to the area of the surface, the factor of proportionality denoted by k being a characteristic of the material where the crack is located. In the present 1D context, the bar is assumed homogeneous and the surface energy associated with an admissible displacement field u reads as :

$$\mathcal{E}_s(u) = \sum_{S(u)} kS = kS\ card(S(u)). \tag{5}$$

2. **Barenblatt's model :** As outlined by Griffith himself, the form (5) of the surface energy is valid from a physical point of view only when the distance between the lips of the crack are large with respect to the characteristic atomic distance. In the spirit of what happens at an atomic scale when atomic bonds break, we will assume, following the idea of Barenblatt in [12], that the surface energy depends on the value of the displacement jump, starting from 0 and progressively growing to its effective Griffith's value k when the displacement jump becomes large with respect to the characteristic atomic length. Specifically, we assume in this model that the surface energy takes the following form :

$$\mathcal{E}_s(u) = \sum_{S(u)} S\kappa([\![u]\!](x)), \tag{6}$$

with

$$\kappa(0) = 0, \quad \kappa\ increasing, \quad \kappa(+\infty) = k. \tag{7}$$

Moreover the convexity properties of κ play an essential role in the analysis of stability of the equilibrium states. Here we adopt the most simple assumption, which can also be justified by microscopic considerations on the interactions of atoms, by assuming that κ is three times differentiable and that

$$\kappa' > 0, \quad \kappa'' < 0, \quad \kappa''' > 0. \tag{8}$$

In the analysis of local minima, the derivative of κ at 0 plays a fundamental role. Let us remark that κ has the dimension of an energy per unit surface and that its derivative has the dimension of a stress. So we put

$$\sigma_c = \kappa'(0), \quad \ell = k/\sigma_c, \tag{9}$$

and we will see that σ_c corresponds to the rupture stress and ℓ to an internal length of the material. Let us remark that κ has to be defined only for positive real number, because of the non-penetration condition (3).

1.4 The total energy.

The total energy of the bar associated to an admissible displacement field u will be the sum of its elastic energy and its surface energy, minus (eventually, when forces are prescribed) the potential of the dead loads. Specifically it reads as

$$\mathcal{E}(u) = \mathcal{E}_b(u) + \mathcal{E}_s(u) - \mathcal{F}(u), \tag{10}$$

with

$$\mathcal{E}_b(u) = \frac{1}{2}\int_{-\infty}^{+\infty} ESu'(x)^2 dx, \tag{11}$$

and

$$\text{- for loading 1} \quad : \quad \mathcal{F}(u) = 0, \tag{12}$$

$$\text{- for loading 2} \quad : \quad \mathcal{F}(u) = \pm F u^+(L), \tag{13}$$

$$\text{- for loading 3} \quad : \quad \mathcal{F}(u) = \pm g \int_0^L u(x)\, dx. \tag{14}$$

1.5 Local and global minima.

We are now able to give a precise definition of the stability of the equilibrium states of the bar.

Definition 1 (Global stability). *We say that an admissible displacement u of the bar submitted to one of the three previous loading cases corresponds to a* ***globally stable equilibrium state*** *if the total energy of the bar in this state is less than the total energy of the bar in any admissible state :*

$$\mathcal{E}(u) \leq \mathcal{E}(v), \quad \forall v \text{ admissible}, \tag{15}$$

where we recall that a field is admissible if it is piecewise smooth and it satisfies the kinematic boundary conditions (1)– (2) *and the non-penetration condition* (3).

Definition 2 (Local stability). *We say that an admissible displacement u of the bar submitted to one of the three previous loading cases corresponds to a* ***locally stable equilibrium state*** *if there exists a neighborhood (in the sense of the chosen norm) of u such that the total energy of the bar in this state is less than the total energy of the bar in any other admissible state in this neighborhood :*

$$\exists \delta(u) > 0, \quad \forall v \text{ admissible}, \|v - u\| \leq \delta(u), v \neq u$$
$$\mathcal{E}(u) < \mathcal{E}(v). \tag{16}$$

The reader will note that the notion of stability of a state depends on the type of loading. In other words, the same displacement field may be a stable equilibrium state when the displacement of the ends are prescribed, but not when they are free. We will encounter several examples in the sequel.

1.6 Quasistatic response of the bar

The property of stability of an equilibrium state only depends on the current loading, but not on the whole process of loading. To complete the analyze and to obtain the response of the bar under a loading process, we have to precise what is the succession of states taken by the bar when the load evolves. We will adopt the quasistatic point of view and assume that the bar will take, for each value of the loading parameter (U, F or g following the loading case), the (or one of the) — if any — (locally or globally) stable equilibrium state(s) corresponding to this value of the loading parameter. This assumption will be sometimes stronger because it will be impossible to find a continuous evolution of equilibrium states with the loading parameter[2], what means that in such a situation the dynamical effects should not be ignored. When several solutions are possible, we will choose — when it is possible — that which leads to a continuously dependence of the state with respect to the loading parameter. Moreover, because of the irreversible character of fracture (at least at the macroscopic scale where we are working), we must also propose a path of states compatible with the irreversibility of the fracture. Let us remark at this purpose that a precise statement of the irreversibility is easily available for Griffith's model, see [1], but less straightforward for Barenblatt's model. This difficulty disappears in our three examples of monotone loading and we will simply verify as the irreversibility condition that the set $S(u)$ of the discontinuity points increases with the loading parameter.

2 Griffith's surface energy and global minimum

We first determine the globally stable equilibrium states for each of the three loading cases when the surface energy is given by (5).

2.1 Prescribed displacements.

Proposition 1. *In the case of a compression* ($u^+(L) = -U$)*, the global minimum corresponds to the elastic solution (without fracture) for any value of* U : $u(x) = -Ux/L$ *when* $0 \leq x \leq L$.

Proof. It is well known that the elastic response has the smallest energy among all continuous admissible displacement fields (Theorem of the potential energy). It remains to prove that the energy of a displacement with at least one point of discontinuity is greater than $\mathcal{E}(u) = ESU^2/2L$. For such a field v, by using the identity $a^2 - b^2 = (a-b)^2 + 2b(a-b)$ and

$$v^+(L) = \int_0^L v' dx + \sum_{S(v)} [\![v]\!], \tag{17}$$

we get

$$\begin{aligned}
\mathcal{E}(v) - \mathcal{E}(u) &= \frac{1}{2}\int_{-\infty}^{+\infty} ES(v'^2 - u'^2)dx + \mathcal{E}_s(v) \\
&> \int_{-\infty}^{+\infty} ESu'(v' - u')dx \\
&= ES\frac{U}{L}\sum_{S(v)} [\![v]\!] \\
&> 0,
\end{aligned}$$

the last inequality being due to $U > 0$ and (3). □

[2] In the language of dynamical systems, that corresponds to the situation where a limit point is reached

Proposition 2. *In the case of a traction ($u^+(L) = +U$), the global minimum corresponds to the elastic solution (without fracture) when* $0 \leq U < U_c = \sqrt{\frac{2kL}{E}}$ *and to an unstrained field with one fracture at an arbitrary point x_1 of the bar when $U > U_c$:*

$$\text{For } U < U_c \quad : \quad u(x) = Ux/L, 0 \leq x \leq L \quad (18)$$

$$\text{For } U > U_c \quad : \quad u(x) = \left\{ \begin{array}{ll} 0, & x < x_1 \\ U, & x > x_1 \end{array} \right. . \quad (19)$$

Proof. Let us first consider the case when $U < U_c$. Let $v \neq u$ be an admissible displacement. When v is continuous, we have $\mathcal{E}(v) > \mathcal{E}(u)$ by virtue of the theorem of the potential energy in linear elasticity. When $S(v)$ is not empty we have $\mathcal{E}(v) \geq kS > ESU^2/(2L) = \mathcal{E}(u)$ and thus the required inequality.

Let us now consider the case when $U < U_c$. In that case $\mathcal{E}(u) = kS$ if u is given by (19). Now the elastic response has an energy equal to $ESU^2/2L > \mathcal{E}(u)$. By virtue of the theorem of the potential energy, any continuous admissible has also a greater energy. On the other hand, any field with more than one jump has an energy greater than kS. The fields v with exactly one jump have a surface energy equals to kS. In order that their elastic energy vanishes, they must have $v'(x) = 0$ everywhere (except at the jump point). By virtue of the boundary conditions (1) and (2), they must then have the form (19).

Finally, we can easily verify that, when $U = U_c$, the global minimum of the energy corresponds to the elastic solution or to the unstrained bar with one fracture. □

Equipped with this determination of the globally equilibrium states of the bar, we can deduce the evolution of the bar during the loading process (by assuming that the bar is at each instant in a globally stable equilibrium state). First, in a compressive test, the bar will always respond elastically. On the other hand, in a tensile test, the bar will first (as long as $U < U_c$) behave elastically, then brutally (as soon as $U > U_c$) breaks at one point, this breaking being followed by a complete release of the elastic energy.

2.2 Prescribed surface traction.

Proposition 3. *In the case of a compression ($\mathcal{F}(v) = -Fv^+(L)$), the global minimum corresponds to the elastic solution (without fracture) for any value of F : $u(x) = -\dfrac{F}{ES}\, x$, $0 \leq x \leq L$.*

Proof. As in the previous case, the elastic solution has an energy less than the energy of any continuous admissible field. Let v be an admissible field with $S(v)$ non empty. We always have

$$\begin{aligned} \mathcal{E}_b(v) - \mathcal{E}_b(u) \quad &\geq \quad ES \int_{-\infty}^{+\infty} u'(v' - u')\, dx \\ &= \quad F\Big(\sum_{S(v)\setminus\{L\}} [\![v]\!] - v^-(L) + u^-(L) \Big). \end{aligned}$$

Since $\mathcal{E}_s(v) > 0$, we deduce that $\mathcal{E}(v) - \mathcal{E}(u) > F \sum_{S(v)} [\![v]\!] > 0$, that is the expected inequality. □

Proposition 4. *In the case of a traction ($\mathcal{F}(v) = Fv^+(L)$), a global minimum does not exist , the energy is not bounded from below.*

Proof. Let v the field such that $v(x) = 0$ for $x < L/2$ and $v(x) = U > 0$ for $x > L/2$. This field is admissible, its elastic energy vanishes, its surface energy equals kS and then its total energy is $kS - FU$. Since U can be taken arbitrarily large, the result follows. □

2.3 Prescribed body forces.

Proposition 5. *In the case of a compression ($\mathcal{F}(v) = -g \int_0^L v(x)\ dx$), the global minimum corresponds to the elastic solution (without fracture) for any value of F : $u(x) = -\dfrac{g}{2ES}\, x(2L-x)$, $0 \leq x \leq L$.*

Proof. As in the previous case, the elastic solution has an energy less than the energy of any continuous admissible field. Let v be an admissible field with $S(v)$ non empty. We have

$$\begin{aligned} \mathcal{E}_b(v) - \mathcal{E}_b(u) &\geq ES \int_{-\infty}^{+\infty} u'(v' - u')\, dx \\ &= -ES \int_0^L u''(v - u) dx - \sum_{S(v)\setminus\{L\}} ESu'[\![v]\!] \\ &= g \int_0^L (v - u) dx + g \sum_{x \in S(v)} (L - x)[\![v]\!](x). \end{aligned}$$

Since $\mathcal{E}_s(v) > 0$, we deduce that $\mathcal{E}(v) - \mathcal{E}(u) > g \sum_{x \in S(v)} (L - x)[\![v]\!](x) \geq 0$, that is the expected inequality. □

Proposition 6. *In the case of a traction ($\mathcal{F}(v) = g \int_0^L v\ dx$), a global minimum does not exist, the energy is not bounded from below.*

Proof. Let v the field such that $v(x) = 0$ for $x < L/2$ and $v(x) = U > 0$ for $x > L/2$. This field is admissible, its elastic energy vanishes, its surface energy equals kS and then its total energy is $kS - gUL/2$. Since U can be taken arbitrarily large, the result follows. □

3 Griffith's surface energy and local minima

3.1 Stability of the elastic response

Proposition 7. *In each case of loading, for any value of the loading parameter, the elastic response is a locally stable equilibrium state.*

Proof. Let us recall the variational property of the elastic response u. In any case, the following equality holds :

$$\int_0^L ESu'\varphi'\,dx = \mathcal{F}(\varphi), \forall \varphi \in \mathcal{D}, \tag{20}$$

where $\mathcal{D}$ denotes the set of differentiable fields such that $\varphi(0) = 0$ and eventually (really, in the first case of loading) $\varphi(L) = 0$. Let us now consider admissible displacements of the form $v = u + h\varphi$, with φ piecewise smooth, verifying the required boundary conditions ($\varphi^-(0) = 0$ and eventually $\varphi^+(L) = 0$), and with norm 1, $\|\varphi\| = 1$. Since $S(u)$ is empty, $S(v) = S(\varphi)$ and we have

$$\begin{aligned}\mathcal{E}(v) - \mathcal{E}(u) &= kS\,card(S(\varphi)) \\ &+ h\left(\int_0^L ESu'\varphi'\,dx - \mathcal{F}(\varphi)\right) \\ &+ \frac{h^2}{2}\int_0^L ES\varphi'^2\,dx.\end{aligned}$$

If $S(\varphi)$ is not empty, then for h small enough we have $\mathcal{E}(v) > \mathcal{E}(u)$, while, when $S(\varphi)$ is empty, by using (20) the linear term in h vanishes and we also get $\mathcal{E}(v) > \mathcal{E}(u)$. □

3.2 Stability of broken responses

Proposition 8. *In the cases of (non null) prescribed surface or body forces, the elastic response is the unique locally stable equilibrium state.*

Proof. We know, by the previous Proposition, that the elastic response is a local minimum. It remains to prove that it is unique. It is clear, by the theorem of the potential energy, that the candidates must be searched among the discontinuous admissible fields (the elastic response is already the unique continuous local minimum among the class of continuous fields). Assume that u is a local minimum with $S(u)$ non empty. Let φ be a piecewise smooth field, verifying the required boundary conditions ($\varphi^-(0) = 0$ and eventually $\varphi^+(L) = 0$), $[\![\varphi]\!] > 0$ on $S(\varphi)\backslash S(u)$ and with norm 1 ($\|\varphi\| = 1$). Put $v = u + h\varphi$ with $h > 0$. For h small enough, v is an admissible field and we must have $\mathcal{E}(v) > \mathcal{E}(u)$ and thus

$$\begin{aligned}0 &< kS\big(card(S(u + h\varphi)) - card(S(u))\big) \\ &+ h\left(\int_0^L ESu'\varphi'\,dx - \mathcal{F}(\varphi)\right) \\ &+ \frac{h^2}{2}\int_0^L ES\varphi'^2\,dx.\end{aligned}$$

Choosing φ such that $S(\varphi) \subset S(u)$, for h small enough we get $S(v) = S(u)$ and $[\![v]\!] > 0$ on $S(u)$ whatever the sign of $[\![\varphi]\!]$ is. Dividing the previous inequality by h and passing to the limit when h goes to 0, we obtain

$$\int_0^L ESu'\varphi'\,dx \geq \mathcal{F}(\varphi), \tag{21}$$

which holds for all φ verifying the boundary conditions and such that $S(\varphi) \subset S(u)$. (The condition $\|\varphi\| = 1$ can be dropped, because (21) remains valid by multiplying φ by $\lambda > 0$.)

Let us finish the proof in the case 2 of loading (the proof of the case 3 is similar). Since the sign of the force does not play a role, it suffices to consider the case of a traction. Now (21) reads as

$$\int_0^L ESu'\varphi'\,dx \geq F\varphi^+(L). \tag{22}$$

Taking φ in $\mathcal{D}$, that is $S(\varphi)$ empty, we first find

$$ESu' = F \quad on \quad [0, L]\backslash S(u).$$

Inserting this relation in (22) leads to

$$F\sum_{S(u)}[\![\varphi]\!] \leq 0,$$

which has to be satisfied by any φ verifying the boundary conditions and such that $S(\varphi) \subset S(u)$. Since the sign of $[\![\varphi]\!]$ is arbitrary on $S(u)$, it is clearly impossible except when $S(u)$ is empty or $F = 0$. □

Proposition 9. *In the cases of prescribed displacements, the responses with an arbitrary finite set of points of fracture separating unstrained parts of the bar are the unique non elastic locally stable equilibrium states.*

Proof. Assume that u is a local minimum with $S(u)$ non empty. Following the proof of the previous proposition, we obtain the inequality (now, $\mathcal{F}(v) = 0$)

$$\int_0^L ESu'\varphi'\,dx \geq 0, \tag{23}$$

which holds for all φ verifying the boundary conditions and such that $S(\varphi) \subset S(u)$. We deduce that it is possible only if $u' = 0$ on $[0, L]\backslash S(u)$, $S(u)$ being arbitrary. Such displacements correspond to those describe in the statement of the proposition. Let us now prove that they are effectively local minima.

Let u be such a displacement, let φ be a piecewise smooth field, verifying the required boundary conditions ($\varphi^-(0) = 0$ and $\varphi^+(L) = 0$), $[\![\varphi]\!] > 0$ on $S(\varphi)\backslash S(u)$ and with norm 1 ($\|\varphi\| = 1$). Put $v = u+h\varphi$ with $h > 0$. For h[3] small enough, v is an admissible field. We get

$$\begin{aligned}\mathcal{E}(v) - \mathcal{E}(u) &= kS\big(card(S(u + h\varphi)) - card(S(u))\big) \\ &+ \frac{h^2}{2}\int_0^L ES\varphi'^2\, dx.\end{aligned}$$

For h small enough, we have $S(u + h\varphi) \supset S(u)$, hence $card(S(u + h\varphi)) \geq card(S(u))$ and the result follows. □

4 Barenblatt's surface energy and local minima

We adopt now the form (6) of the surface energy. On the first hand its dependence on the value of the jump of the displacement leads to a less straightforward analysis, in particular the determination of the global minima — when they exist — pass by that of the local ones. On the other hand the energy is now a differentiable function of the displacement. To characterize the local minima we proceed in two steps :

1. We first establish the so-called first order necessary conditions of local stability which lead to the notion of *equilibrium states*.

2. We then determine those which are stable by the mean of second order sufficient conditions.

4.1 Equilibrium states

Let u be a local minimum, φ a piecewise smooth field, verifying the required boundary conditions ($\varphi^-(0) = 0$ and eventually $\varphi^+(L) = 0$), $[\![\varphi]\!] > 0$ on $S(\varphi)\backslash S(u)$ and with norm 1 ($\|\varphi\| = 1$). Put $v = u + h\varphi$ with $h > 0$ and small enough in order that v is admissible and that $S(v) \supset S(u)$. Moreover since $\mathcal{E}(u + h\varphi) > \mathcal{E}(u)$, by dividing by h and passing to the limit when h tends to 0, we obtain the following inequality, called *the first order necessary stability condition* :

$$\mathcal{E}'(u)(\varphi) \geq 0,$$

[3]To complete the proof, we should show that h can be chosen independently of φ. That leads to too technical developments for this short paper.

which explicitly reads as

$$\begin{aligned} 0 \leq& \int_0^L ESu'\varphi' dx - \mathcal{F}(\varphi) \\ &+ \sum_{S(u)} \kappa'([\![u]\!])S[\![\varphi]\!] + \sum_{S(\varphi)\backslash S(u)} \sigma_c S[\![\varphi]\!], \quad (24)\end{aligned}$$

which has to be satisfied by any admissible φ. Inequalities (24) are only necessary conditions of stability. A field u that satisfies it is not necessarily a local minimum, we will call it *an equilibrium state.*

Let us remark that by taking φ in $\mathcal{D}$, since $\mathcal{D}$ is a linear space, we deduce from (24) that an equilibrium state satisfies also

$$\int_0^L ESu'\varphi' dx = \mathcal{F}(\varphi), \quad \forall\varphi \in \mathcal{D}. \qquad (25)$$

We now establish the set of local conditions satisfied by an equilibrium state for each case of loading.

Proposition 10. *In the case of prescribed displacements, a piecewise smooth field u is an equilibrium state if and only if it satisfies the following conditions :*

$$ESu'(x) = F \quad in \quad [0, L]\backslash S(u) \qquad (26)$$
$$\kappa'([\![u]\!])S = F \quad on \quad S(u) \qquad (27)$$
$$F \leq \sigma_c S \qquad (28)$$

where F is a (unknown) constant, and

$$[\![u]\!] \geq 0\, on\ S(u), \quad u^-(0) = 0, \quad u^+(L) = \pm U. \quad (29)$$

Proof. The relations (29) are simply the kinematic conditions. Since $\mathcal{F} = 0$, (26) is equivalent to (25). Inserting it in (24) leads to

$$0 \leq \sum_{S(u)} (\kappa'([\![u]\!])S - F)[\![\varphi]\!] + \sum_{S(\varphi)\backslash S(u)} (\sigma_c S - F)[\![\varphi]\!].$$

Since the sign of $[\![\varphi]\!]$ is arbitrary on $S(u)$ but positive on $S(\varphi)\backslash S(u)$, this last inequality is equivalent to (27)-(28), what completes the proof. □

By proceeding in the same way, we obtain the required conditions in the two latter cases of loading.

Proposition 11. *In the case of prescribed surface force, a piecewise smooth field u is an equilibrium state if and only if it satisfies the following conditions :*

$$ESu'(x) = \pm F \quad in \quad [0, L]\backslash S(u) \qquad (30)$$
$$\kappa'([\![u]\!])S = \pm F \quad on \quad S(u) \qquad (31)$$
$$\pm F \leq \sigma_c S \qquad (32)$$

where F is the (given) intensity of the applied force, and

$$[\![u]\!] \geq 0\, on\ S(u), \quad u^-(0) = 0. \qquad (33)$$

Proposition 12. *In the case of prescribed body forces, a piecewise smooth field u is an equilibrium state if and only if it satisfies the following conditions :*

$$ESu'(x) = \pm g(L-x) \quad in \quad [0,L]\setminus S(u) \quad (34)$$
$$\kappa'([\![u]\!](x))S = \pm g(L-x) \quad on \quad S(u) \quad (35)$$
$$\pm gL \leq \sigma_c S \quad (36)$$

and

$$[\![u]\!] \geq 0 \, on \; S(u), \quad u^-(0) = 0. \quad (37)$$

We can remark that these first order necessary conditions of stability contain not only the usual equilibrium equations of the bar under the prescribed loading but also a yield condition on the stress field, cf (28), (32) and (36). It is probably the most important change given by Barenblatt' model.

We can also note that the elastic response verifies these conditions provided that the loading parameter is not larger than a critical value. Specifically it is easily checked that the elastic response is an equilibrium state (in the present sense)

1. for $U \leq \dfrac{\sigma_c}{E}L$, in the case 1 with traction $(+U)$;
2. for $F \leq \sigma_c S$, in the case 2 with traction $(+F)$;
3. for $g \leq \dfrac{\sigma_c S}{L}$, in the case 3 with traction $(+g)$;

while they are always equilibrium states in compression.

Moreover the elastic response is the unique equilibrium state in compression. Indeed, if there exists other equilibrium states u, they must suffer at least one jump. But since $\kappa' > 0$, it is impossible in the two latter cases, by virtue of (31) or (35). In the first case, we should have $F > 0$ and hence $u' > 0$. But, from the kinematic conditions we obtain $-U = \int_0^L u'dx + \sum_{S(u)} [\![u]\!] > 0$, that is a contradiction.

On the other hand, in the traction cases, many other equilibrium states exist. Let us first consider the cases 2 and 3. We obtain

Proposition 13. *In the case of a prescribed surface traction, at each value of F such that $0 < F < \sigma_c S$, we can associate an infinite family of equilibrium states indexed by their set of discontinuity points. Specifically, let $\mathcal{S}$ be a finite subset of $[0,L]$, then u defined by $u^-(0) = 0$, $u' = F/S$ on $[0,L]\setminus\mathcal{S}$ and $[\![u]\!] = (\kappa')^{-1}(F/S)$ on $\mathcal{S}$ is an equilibrium state.*

Proposition 14. *In the case of prescribed tensile body forces, at each value of g such that $0 < gL < \sigma_c S$, we can associate an infinite family of equilibrium states indexed by their set of discontinuity points. Specifically, let $\mathcal{S}$ be a finite subset of $[0,L]$, then u defined by $u^-(0) = 0$, $u'(x) = g(L-x)/ES$ on $[0,L]\setminus\mathcal{S}$ and $[\![u]\!](x) = (\kappa')^{-1}(g(L-x)/S)$ on $\mathcal{S}$ is an equilibrium state.*

Since the proofs are straightforward from the previous characterizations of the equilibrium states, we omit them. On the other hand, the case 1 needs a more careful presentation.

Proposition 15. *In the case of a prescribed tensile displacement, at each value of F such that $0 < F < \sigma_c S$ and each finite subset $\mathcal{S}$ of $[0,L]$, we can associate an equilibrium state u (defined by $u^-(0) = 0$, $u' = F/S$ on $[0,L]\setminus\mathcal{S}$ and $[\![u]\!] = (\kappa')^{-1}(F/S)$ on $\mathcal{S}$) corresponding to a prescribed tensile displacement the value U of which is related to F by*

$$U = \frac{FL}{ES} + card(\mathcal{S})(\kappa')^{-1}(F/S). \quad (38)$$

Proof. The first part of the statement is clear. Equation (38) follows from the relation $U = u^+(L) = \int_0^L u'dx + \sum_{\mathcal{S}} [\![u]\!]$ and the fact that u' and $[\![u]\!]$ are constant. □

We can note that for a given U the number of equilibrium states is not obvious, but depends on the property of the function κ' and on the ratio ℓ/L between the material length and the bar length. On the other hand, in the plane (F,U) they correspond to a countable family of curves indexed by $card(\mathcal{S})$, starting from the "bifurcation point" $(\sigma_c S, L\sigma_c/E)$ and finishing to $(0,+\infty)$. These branches of equilibrium can be parametrized by F, U is then a function of F. The branch associated with $\mathcal{S} = \emptyset$ is nothing but the elastic branch : $U = FL/ES$, while that associated with $n = card(\mathcal{S}) > 0$ corresponds to the equilibrium of the bar with n cuts arbitrarily distributed. On these "fracturing branches" U is a convex function of F, say $U_n(F)$, by virtue of the assumed convexity properties (8) of the surface energy. Since

$$U'_n(0) = \frac{L}{ES} + \frac{n}{\kappa''(0)S},$$

the n^{th}-branch is monotone (U_n decreasing) provided that the length of the bar is small enough, specifically when

$$L \leq L_n = \frac{nE}{|\kappa''(0)|}.$$

In particular, when $L \leq L_1$ all the branches are decreasing from ∞ to $L\sigma_c/E$. Otherwise, when $L > L_1$, the first branches are first decreasing, then increasing and pass by a minimum, see the Figure 1 in which the elastic branch and the first branch U_1 are represented.

It remains to find what equilibrium states are stable. It is the goal of the next subsection.

4.2 Locally stable equilibrium states

Let u be an equilibrium state, φ a piecewise smooth field, verifying the required boundary conditions ($\varphi^-(0) = 0$ and eventually $\varphi^+(L) = 0$), $[\![\varphi]\!] > 0$ on $S(\varphi)\setminus S(u)$) and with norm 1 ($\|\varphi\| = 1$). Put

$v = u + h\varphi$ with $h > 0$ and small enough in order that it is admissible and that $S(v) \supset S(u)$. Then $\mathcal{E}$ is twice differentiable at u in the direction φ and by developing it up to the second order we obtain

$$\mathcal{E}(u + h\varphi) - \mathcal{E}(u) = h\mathcal{E}'(u)(\varphi) + \frac{h^2}{2}\mathcal{E}''(u)(\varphi) + o(h^2). \quad (39)$$

Since, by definition, u satisfies $\mathcal{E}'(u)(\varphi) \geq 0$, we will have $\mathcal{E}(u + h\varphi) > \mathcal{E}(u)$ for h sufficiently small in any direction such that $\mathcal{E}'(u)(\varphi) > 0$. So we have only to consider the directions such that $\mathcal{E}'(u)(\varphi) = 0$. The analysis of the previous subsection shows that they correspond to the (admissible) directions φ such that $S(\varphi) \subset S(u)$[4]. For such directions (39) becomes $\mathcal{E}(u + h\varphi) = \mathcal{E}(u) + \frac{h^2}{2}\mathcal{E}''(u)(\varphi) + o(h^2)$ and we will have $\mathcal{E}(u + h\varphi) > \mathcal{E}(u)$ for h sufficiently small provided that $\mathcal{E}''(u)(\varphi) > 0$. On the other hand, if $\mathcal{E}''(u)(\varphi) < 0$, we will have $\mathcal{E}(u + h\varphi) < \mathcal{E}(u)$ for h sufficiently small and u is not a local minimum. We have thus obtained the second order conditions of stability of an equilibrium state.

Proposition 16.
-Second order necessary stability conditions. *An equilibrium state is locally stable* only if $\mathcal{E}''(u)(\varphi) \geq 0$ *for any piecewise smooth* φ *such that* $S(\varphi) \subset S(u)$, $\varphi^-(0) = 0$ *(and eventually* $\varphi^+(L) = 0$*).*

-Second order sufficient stability conditions. *An equilibrium state is locally stable* if $\mathcal{E}''(u)(\varphi) > 0$ *for any piecewise smooth* $\varphi \neq 0$ *such that* $S(\varphi) \subset S(u)$, $\varphi^-(0) = 0$ *(and eventually* $\varphi^+(L) = 0$*).*

Since the second derivative of the energy in the examined directions reads as

$$E''(u)(\varphi) = \int_0^L ES\varphi'^2 dx + \sum_{S(u)} S\kappa''([\![u]\!])[\![\varphi]\!]^2, \quad (40)$$

we immediately obtain

Proposition 17. *The elastic equilibrium states (that is the elastic responses of the bar when the loading parameter is smaller than its critical value) are locally stable.*

Proof. Since $S(u)$ is empty, the result follows from (40) and the second order sufficient stability condition. □

When the forces are prescribed, the elastic equilibrium states are the unique locally stable states as the following Proposition proves it.

Proposition 18. *In the loading cases 2 and 3, any equilibrium state with jump discontinuities is unstable.*

Proof. Let u be an equilibrium state and $x_1 \in S(u)$ a point of discontinuity. Put $\varphi(x) = 0$ for $x < x_1$ and $\varphi(x) = 1$ for $x > x_1$. This field is an admissible direction with $S(\varphi) \subset S(u)$. But, since $\varphi' = 0$, we have $E''(u)(\varphi) = S\kappa''([\![u]\!](x_1)) < 0$ and the result follows by virtue of the second order necessary stability conditions. □

It remains to study the stability of the inelastic equilibrium states in the first loading case where the displacements are prescribed at both ends of the bar. That leads to the following Proposition :

Proposition 19. *In the loading case 1, any equilibrium state with more than one discontinuity point is unstable. Only the equilibrium states with exactly one discontinuity point and located on the decreasing part of the branch* U_1 *are locally stable.*

Proof. Let u be an equilibrium state with $n > 1$ points of discontinuity, $S(u) = \{x_1, \cdots, x_n\}$. Put $\varphi(x) = 0$ for $x < x_1$ or $x > x_n$ and $\varphi(x) = 1$ for $x_1 < x < x_n$. This field is an admissible direction with $S(\varphi) \subset S(u)$ (let us recall that the sign of $[\![\varphi]\!]$ can be chosen arbitrarily on $S(u)$). But, since $\varphi' = 0$, we have $E''(u)(\varphi) = S(\kappa''([\![u]\!](x_1) + \kappa''([\![u]\!](x_n)) < 0$ and the first part of the Proposition follows by virtue of the second order necessary stability conditions.

Now, let u be an equilibrium state with 1 point of discontinuity, $S(u) = \{x_1\}$. Then (40) becomes

$$E''(u)(\varphi) = \int_0^L ES\varphi'^2 dx + S\kappa''([\![u]\!](x_1))[\![\varphi]\!]^2.$$

Let us introduce

$$\lambda = \min\left\{\int_0^L \varphi'^2 dx \mid \varphi \in \mathcal{S}_1\right\}, \quad (41)$$

where $\mathcal{S}_1$ denotes the set of fields φ smooth on $\mathbb{R}\backslash\{x_1\}$, such that $\varphi = 0$ on $\mathbb{R}\backslash[0, L]$ and $[\![\varphi]\!](x_1) = 1$..

Owing to the second order necessary and sufficient stability conditions, u is stable provided that $\lambda E + \kappa''([\![u]\!](x_1)) > 0$ and unstable if $\lambda E + \kappa''([\![u]\!](x_1)) < 0$. Classical tools of Variational Calculus give us that the minimizer φ^* giving λ is the element of $\mathcal{S}_1$ such that $\int_0^L \varphi^{*\prime}\varphi' dx = 0$, for all φ in $\mathcal{D}$. Straightforward calculations give then $\varphi^*(x) = -x/L$ for $x < x_1$ and $\varphi^*(x) = 1 - x/L$ for $x > x_1$. So

$$\lambda = 1/L, \quad (42)$$

and, since $[\![u]\!](x_1) = (\kappa')^{-1}(F/S)$, the stability condition of u now reads as $E + \kappa''((\kappa')^{-1}(F/S))L > 0$. Since

$$U_1'(F) = \frac{L}{ES} + \frac{1}{\kappa''((\kappa')^{-1}(F/S))S}$$

the result follows. □

[4] Except when the critical loading parameter is reached. The determination of the stability of the bifurcation point needs a separate analysis. We omit it here.

4.3 Quasistatic responses of the bar

Equipped with this determination of the locally equilibrium states of the bar, we can deduce the evolution of the bar during the loading process (by assuming that the bar is at each instant in a locally stable equilibrium state).

1. First, in any compressive test, the bar will always respond elastically.

2. In a tensile test with prescribed surface traction (loading 2, $+F$), the bar will first (as long as $F < \sigma_c S$) behave elastically, then (as soon as $F > \sigma_c S$) the equilibrium is impossible and a complete dynamical study must be developed to determine what happens. Let us remark that the critical stress is reached simultaneously at every point of the bar.

3. In a tensile test with prescribed body forces (loading 3, $+g$), the bar will first (as long as $gL < \sigma_c S$) behave elastically, then (as soon as $gL < \sigma_c S$) the equilibrium is impossible and a complete dynamical study must be developed to determine what happens. Let us remark however that the critical stress is reached at the end $x = 0$ of the bar and we can expect that the bar will break here.

4. In a tensile test with prescribed displacement (loading 1, $+U$), if the bar is short enough ($L \leq L_1 = E/\kappa''(0)$), the bar will first (as long as $EU \leq \sigma_c L$) behave elastically, then (as soon as $EU > \sigma_c L$) a cut appears at an arbitrary point x_1, the two equally strained parts of the bar leave each other and the stress progressively decreases to 0. The fracture evolves then smoothly.

5. In a tensile test with prescribed displacement (loading 1, $+U$), if the bar is long enough ($L > L_1 = E/\kappa''(0)$), the bar will first (as long as $EU < \sigma_c L$) behave elastically, then, when $EU = \sigma_c L$, brutally the bar breaks at an arbitrary point x_1, the two parts of the bar separate suddenly and the stress brutally decreases. After, when $EU > \sigma_c L$, the two equally strained parts of the bar continue progressively to leave each other and the stress progressively decreases to 0. The end of the fracture is smooth.

5 Barenblatt's surface energy and global minimum

To complete this comparison between Griffith's and Barenblatt's model, we determine in the present section the global minima of Barenblatt's model, when they exist.

We first consider the cases of compressive loads.

Proposition 20. *In the cases of compressive loads, the elastic responses are the unique globally stable equilibrium states.*

Proof. The proof given for Griffith's model is still valid, because we only used the positivity of the surface energy. □

Let us now consider the cases of tensile applied forces. As for Griffith model, we obtain

Proposition 21. *In the case of applied tensile forces, a global minimum does not exist, the energy is not bounded from below.*

Proof. Let v the field such that $v(x) = 0$ for $x < L/2$ and $v(x) = U > 0$ for $x > L/2$. This field is admissible, its elastic energy vanishes, its surface energy equals $\kappa(U)S < kS$ and then its total energy is less than $kS - FU$ in the case 2 and less than $kS - gUL/2$ in the case 3. Since U can be taken arbitrarily large, the result follows. □

Finally, we consider the case of tensile prescribed displacement.

Proposition 22. *In the case of a prescribed tensile displacement U,*

1. For small bars, when $L \leq L_1$, the elastic response is globally stable as long as $U \leq L\sigma_c/E$, while the equilibrium states of the first branch U_1 are globally stable as soon as $U \geq L\sigma_c/E$.

2. For long bars, when $L > L_1$, the elastic response is globally stable as long as $U \leq U^*$, while the locally stable equilibrium states of the first branch U_1 are globally stable as soon as $U \geq U^*$, where U^* corresponds to the Maxwell displacement (see Figure 2).

Proof. For Griffith's model, we proved directly the results, by verifying that the expected candidate is really a minimizer. In the case of Barenblatt surface energy, we can not use this way. Moreover the rigorous proof of the existence of a minimum required advanced tools of functional analysis. In the present paper, we will simply remark that the energy is bounded from below, omitting the verification that the infimum is effectively reached. Once we know that a global minimum exists, we will find it among the set of local minima. We have to distinguish 2 cases depending on the bar length.

When the bar length is smaller than L_1, then at each value of the loading parameter U is associated one and only one local minimum, see Figure 5. It is then necessary also the global one.

When the bar length is longer than L_1, then, since the branch U_1 is not monotone, there exists an interval in which at each value of the loading parameter corresponds two local minima, see Figures 1 and 4.

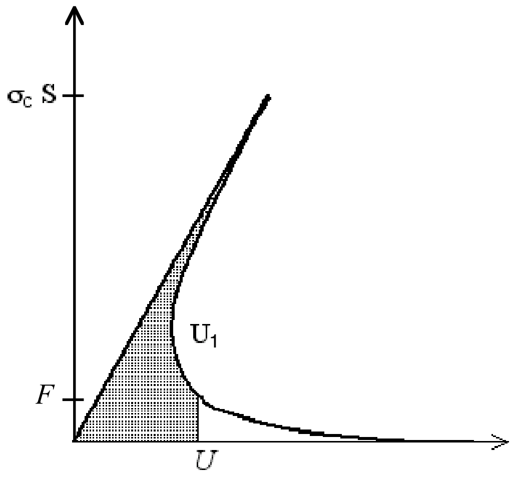

Figure 1: *Graphic interpretation of the global energy of a locally stable equilibrium state containing one cut.*

We have to compare their value. The energy of the elastic equilibrium state is $\mathcal{E}_0 = ESU^2/2L$ while the energy of the "broken" stable equilibrium state is

$$\mathcal{E}_1 = \frac{F^2 L}{2ES} + \kappa((\kappa')^{-1}(F/S))S, \qquad (43)$$

the force F being related to the prescribed displacement U by

$$U = \frac{FL}{ES} + (\kappa')^{-1}(F/S). \qquad (44)$$

This value can be interpreted graphically. Indeed, $\mathcal{E}_1$ is nothing but the colored area on the Figure 1[5].

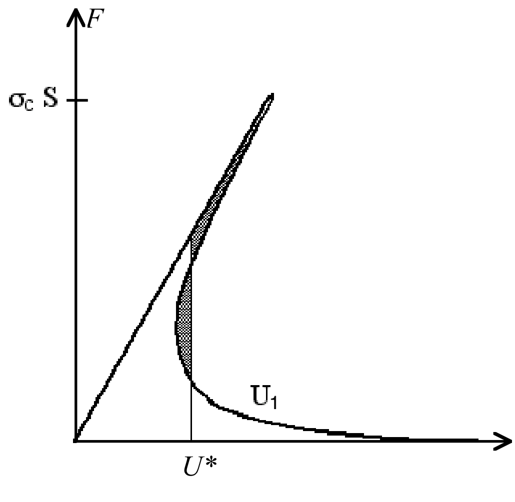

Figure 2: *Graphic determination of the global minimum and the Maxwell line.*

In consequence the determination of the global minimum is graphically straightforward. Indeed, since $\mathcal{E}_0$ is the area of the triangle given by the elastic branch segment line, $\mathcal{E}_0$ will be less (resp. higher) than $\mathcal{E}_1$ when U is less (resp. larger) than U^* which corresponds to the point such that the two colored areas on the Figure 2 are equal. That corresponds to the famous Maxwell rule. □

[5]The verification is given in the appendix.

6 Conclusion

The comparison of Griffith's model with Barenblatt's model is summarized in the three following graphics representing the portraits of the branches of equilibrium states with their stability in the case of prescribed tensile displacements. The stability of a state is indicated by the width of the branch :

▬ : global minimum,

— : local minimum,

___ : unstable equilibrium.

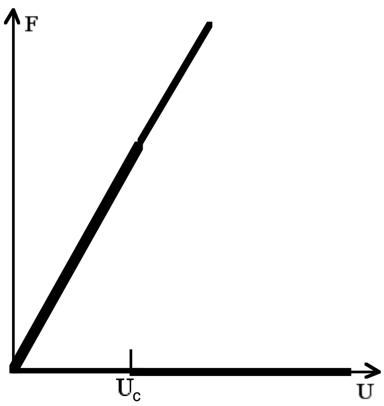

Figure 3: *Griffith's model : Portrait of the branches of equilibrium states and their stability in the case of prescribed tensile displacements.*

The main disadvantage of Griffith's energy surface is that the elastic response always remains locally stable (as well for applied displacements as for applied forces), while Barenblatt's energy surface destabilizes this response by introducing a yield stress σ_c. These properties are proved here in a one-dimensional context, but hold true in higher dimensions as we will show in a paper in preparation.

The main disadvantage of the assumption that the structure only searches for the global minima is first that it does not work when tensile forces are applied because the energy is not bounded from below (it is also a general result, valid in any dimension). Moreover it induces spurious size effects. Indeed, let us

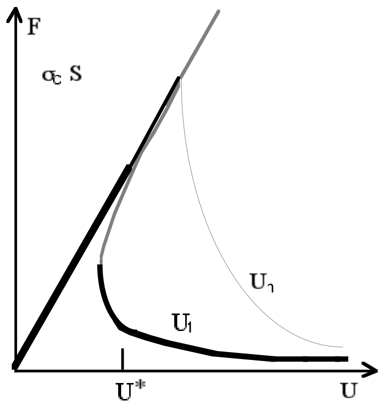

Figure 4: *Barenblatt's model and $L > L_1$: Portrait of the branches of equilibrium states and their stability in the case of prescribed tensile displacements.*

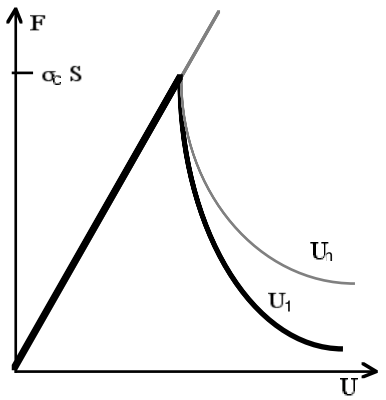

Figure 5: *Barenblatt's model and $L \leq L_1$: Portrait of the branches of equilibrium states and their stability in the case of prescribed tensile displacements.*

consider the case of applied displacements with Griffith'model. We have found that the bar breaks when U reaches the critical value

$$U_c = \sqrt{\frac{2kL}{E}}$$

corresponding to the critical strain $\sqrt{\frac{2k}{EL}}$ and the critical stress $\sqrt{\frac{2kE}{L}}$. Thus, longer is the bar, smaller is the rupture stress, and even, when the bar length tends to infinity, the rupture stress tends to 0. A similar result holds with Barenblatt's surface energy (the verification is left to the reader). By authorizing also local minima, the size effects do not disappear (compare Figure 4 with Figure 5) because both models contain the internal length ℓ. But (at least in this 1D study) the size effects due to local minima and Barenblatt's surface energy are physically satisfactory.

It remains to investigate the properties of local minima for Barenblatt's model in higher dimensions. A major challenge.

References

[1] G. A. Francfort, J.-J. Marigo. Revisiting brittle fracture as an energy minimization problem. *J. Mech. Phys. Solids, 46* **(8)**,1998, 1319–1342.

[2] F. Bilteryst. Une approche énergétique de la décohésion et de la microfissuration dans les composites. *Thèse de doctorat de l'Université Pierre et Marie Curie, 2000, Paris.*

[3] F. Bilteryst, J.-J. Marigo . Amorçage de la décohésion dans l'essai d'arrachement. *C. R. Acad. Sci. Paris, t. 327, Série II* **b**, 1999, 977–983.

[4] B. Bourdin. Une méthode variationnelle en mécanique de la rupture. Théorie et applications numériques. *Thèse de doctorat de l'Université Paris-Nord*, 1998, 178p..

[5] B. Bourdin, G. A. Francfort, J.-J. Marigo. Numerical experiments in revisited brittle fracture. *J. Mech. Phys. Solids, 48* , 2000, 797–826.

[6] L. Truskinovsky. Fracture as a phase transition. in *Contemporary Research in the Mechanics and Mathematics of Materials*, ed. r. c. Batra and m.f. Beatty, CIMNE, Barcelona, 1996, 322–332.

[7] G. Del Piero. One dimensional ductile-brittle transition, yielding, and structured deformations. in *Proceedings IUTAM Symposium "Variations de domaines et frontières libres en mécanique"*, eds. p. Argoul and m. Frémond, Paris, 1997, (Kluwer, 1999).

[8] G. Del Piero, L. Truskinovsky. Macro- and micro-cracking in one-dimensional elasticity. *Int. J. Solids Struct.*, 2000, (in press).

[9] A. Kolmogorov, S. Fomine. *Eléments de la théorie des fonctions et de l'analyse fonctionnelle.* Editions Mir-Moscou, 1974.

[10] A. Braides, G. Dal Maso, A. Garroni . Variational formulation of softening phenomena in fracture mechanics : the one dimensional case. *Arch. Rat. Mech. Anal.*, 1999.

[11] A. Griffith. The phenomena of rupture and flow in solids. *Phil. Trans. Roy. Soc. London CCXXI-A*, 1920, 163–198.

[12] G. I. Barenblatt. The mathematical theory of equilibrium cracks in brittle fracture. *Adv. Appl. Mech., 7* , 1962, 55–129.

APPENDIX : The graphic interpretation of the energy

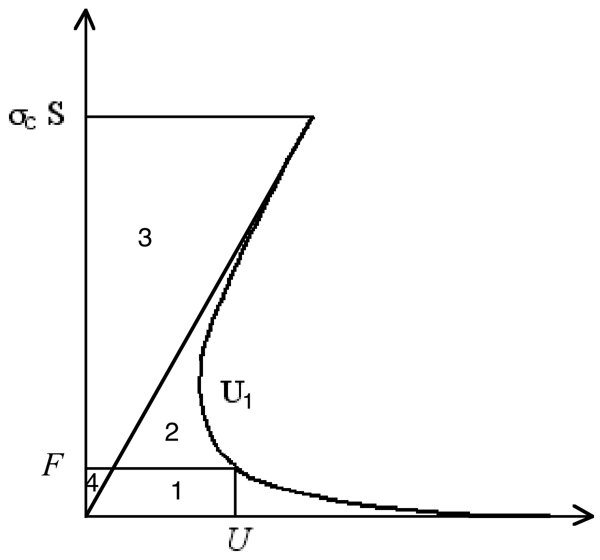

Figure 6: *Graphic calculation of the energy.*

We have to prove that in the case of a prescribed displacement U the energy $\mathcal{E}_1$ of the locally stable equilibrium field u located on the decreasing part of the branch U_1 corresponds to the darked area $\mathcal{A}$ on the Figure 1.

Let us consider Figure 6 with the four numbered areas $\mathcal{A}_i$, $i = 1, 4$. We easily check the following relations

$$\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2, \quad (45)$$

$$\mathcal{A}_4 = \frac{F^2 L}{2ES}, \quad (46)$$

$$\mathcal{A}_1 + \mathcal{A}_4 = FU, \quad (47)$$

$$\mathcal{A}_3 + \mathcal{A}_4 = \frac{\sigma_c^2 SL}{2E}, \quad (48)$$

$$\mathcal{A}_2 + \mathcal{A}_3 = \int_F^{\sigma_c S} U_1(f) df. \quad (49)$$

From (44), we obtain

$$\begin{aligned}\mathcal{A}_2 + \mathcal{A}_3 &= \frac{\sigma_c^2 SL}{2E} - \frac{F^2 L}{2ES} + \int_F^{\sigma_c S} \kappa'^{-1}(f/S) df \\ &= \mathcal{A}_3 + \int_F^{\sigma_c S} \kappa'^{-1}(f/S) df,\end{aligned}$$

what yields

$$\mathcal{A}_2 = \int_F^{\sigma_c S} \kappa'^{-1}(f/S) df. \quad (50)$$

Introducing the change of variable $\delta = \kappa'^{-1}(f/S)$ gives

$$\mathcal{A}_2 = -\int_0^{U - \frac{FL}{ES}} \delta \kappa''(\delta) S \, d\delta. \quad (51)$$

A direct calculation of the latter integral leads

$$\mathcal{A}_2 = -FU + \frac{F^2 L}{ES} + \kappa\Big(\kappa'^{-1}(F/S)\Big) S \quad (52)$$

and finally

$$\mathcal{A} = \frac{F^2 L}{2ES} + \kappa\Big(\kappa'^{-1}(F/S)\Big) S,$$

what exactly corresponds to $\mathcal{E}_1$, see (43).