Gradient damage models: construction and fundamental properties

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Quiberon, 22-27 september 2014

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"Analyse Variationnelle et Microstructuration"

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Notation

Throughout these Lecture Notes the following notation is used:

- In n-dimension, $n > 1$,the vectors and second order tensors are indicated by boldface letters, like $\mathbf{u}$, $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$ for the displacement vector, the strain tensor and the stress tensor. Their components are denoted by italic letters, like u_i and σ_{ij}.
- The reference configuration of a material point is $\mathbf{x}$ and its cartesian coordinates in $\mathbb{R}^n$ are $(x_1, \cdots, x_n)$. The orthonormal basis of $\mathbb{R}^n$ is $(\mathbf{e}_1, \cdots, \mathbf{e}_n)$, $\mathbb{M}^n_s$ denotes the space of 2×2 symmetric tensors of $\mathbb{R}^n$ and $\mathbf{I}$ is its identity tensor.
- The fourth order tensors as well as their components are indicated by a sans serif letter, like E or E_{ijkl} for the stiffness tensor. Such tensors are considered as linear maps applying on vectors or second order tensors and the application is denoted without dots, like $\mathsf{E}\boldsymbol{\varepsilon}$ whose ij-component is $\mathsf{E}_{ijkl}\varepsilon_{kl}$. The summation convention on repeated indices is implicitly adopted. The inner product between two vectors or two tensors of the same order is indicated by a dot, like $\mathbf{a} \cdot \mathbf{b}$ which stands for $a_i b_i$ or $\boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}$ for $\sigma_{ij}\varepsilon_{ij}$.
- The symbol $\otimes$ denotes the tensor product and $\otimes_s$ its symmetrized, *i.e.* $2\mathbf{e}_1 \otimes_s \mathbf{e}_2 = \mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1$.
- In one-dimension, $n = 1$, all the scalar quantities or fields are indicated by italic letters, like u, ε, σ or $E(\alpha)$ for the displacement, the strain, the stress or the damaged Young modulus. The prime denotes either the derivative with respect to the coordinate x or the derivative with respect to the damage parameter, e.g. $u' = \partial u/\partial x$, $E'(\alpha) = dE(\alpha)/d\alpha$
- The dot stands for the time derivative, e.g. $\dot{\alpha} = \partial\alpha/\partial t$.
- The qualifier *increasing* (resp. *decreasing*) stands for *strictly increasing* (resp. *strictly decreasing*) and should not be confused with *non decreasing* (resp. *non increasing*). In the same way, the qualifier *positive* (resp. *negative*) stands for > 0 (resp. < 0) and not for ≥ 0 (resp. ≤ 0).
- The classical convention is adopted for the orders of magnitude: $o(h)$ denotes functions of h such that $\lim_{h\to 0} o(h)/h = 0$.

Part I

Construction of damage models

Abstract

The first part of these Lecture Notes is devoted to the construction of brittle damage laws. Starting from the general class of local damage laws based on the concept of yield criterion, we first justify from Drucker-Ilyushin postulate that these laws can be formulated within the framework of Generalized Standard Materials developed by Nguyen (2000). Accordingly, the strain work becomes a state function whose convexity properties are directly related to the hardening or softening properties of the material. Moreover, the evolution problem can automatically be read as a variational problem (Pham and Marigo, 2010a). This naturally appearing formulation is reinforced so that it finally contains the concepts of stability and energy conservation (Mielke, 2005; Bourdin et al., 2008). In the case of softening materials, it turns out that these type of models lead to ill-posed mathematical problems (Comi, 1995) because of their local character and the absence of terms limiting the damage localization. To avoid these pathological localizations we introduce *gradient of damage* terms in the model which contains accordingly (at least) one *characteristic* length. Then, following Pham and Marigo (2010b), the evolution problem associated with this enhanced model is obtained by using the principles of irreversibility, stability and energy balance.

1. Construction of local brittle damage models

1.1. The fundamental ingredients

If one follows the procedure proposed by Marigo (1981), the construction of a damage model consists in the three following steps:

(i) choice of the damage parameter;

(ii) choice of the dependence of the constitutive stress-strain relation on this parameter;

(iii) choice of the law which governs the evolution of the damage parameter.

Our goal is not to develop a model for a precise application but rather to give a general framework for constructing any brittle damage model. Accordingly, in this introductory lecture we will make the simplest choices so that to emphasize the fundamental concepts. Specifically we assume that:

(i) The damage state of a material point can be described by a scalar α. This is of course a very strong assumption which is essentially justified by a sake of simplicity, these types of phenomenological models being devoted to the computation of mechanical structures. The choice of the parameter and consequently its physical interpretation remain at this stage arbitrary. Starting from a first choice, we will always have the possibility to make a change of variable. This opportunity will be used in the sequel. Therefore, at this stage, we merely assume that α grows from 0 to α_m where $0 < \alpha_m \leq +\infty$, 0 corresponds to the undamaged state and α_m to the full damaged state;

(ii) At given α, the material has an elastic material. Its elasticity depends on the damage variable through the elastic potential $\psi(\boldsymbol{\varepsilon},\alpha)$ which is assumed to be continuously differentiable in $\mathbb{M}^n_s \times [0,\alpha_m)$, $\mathbb{M}^n_s$ denoting the space of second order tensors and n the space dimension. To simplify the presentation we will assume that ψ is a quadratic function of $\boldsymbol{\varepsilon}$ at given α, *i.e.*

$$\psi(\boldsymbol{\varepsilon},\alpha) = \tfrac{1}{2}\mathsf{E}(\alpha)\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}, \tag{1}$$

where $\mathsf{E}(\alpha)$ denotes the fourth order stiffness tensor. Accordingly, the material behavior is linearly elastic at given α and the stress-strain relation reads as

$$\boldsymbol{\sigma} = \frac{\partial\psi}{\partial\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon},\alpha) = \mathsf{E}(\alpha)\boldsymbol{\varepsilon}. \tag{2}$$

The fact that α really corresponds to a damage variable results in a decrease of the stiffness when α grows. Therefore, we will assume that the function $\alpha \mapsto \mathsf{E}(\alpha)$ satisfies the following properties:

$$\mathsf{E}(0) > 0, \qquad \mathsf{E}'(\alpha) < 0, \qquad \mathsf{E}(\alpha_m) = 0. \tag{3}$$

The inequalities above must be understood in the sense of the positivity of fourth order tensors. Specifically, a fourth order tensor A is said positive when the following inequality holds:

$$\mathsf{A}\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon} > 0, \quad \forall\boldsymbol{\varepsilon}\in\mathbb{M}^n_s, \boldsymbol{\varepsilon}\neq 0.$$

Therefore, we assume in (3) that the material progressively looses its rigidity and has no more rigidity when it is fully damaged. As long as $\alpha < \alpha_m$, the stiffness tensor $\mathsf{E}(\alpha)$ is positive and hence invertible. Its inverse is the compliance tensor $\mathsf{S}(\alpha)$,

$$\mathsf{S}(\alpha) = \mathsf{E}(\alpha)^{-1} \tag{4}$$

which gives the strain in terms of the stress for a given damage state,

$$\boldsymbol{\varepsilon} = \mathsf{S}(\alpha)\boldsymbol{\sigma}. \tag{5}$$

(iii) For irreversibility reasons, damage can only grow and its growth is governed by a yield criterion (like in plasticity). Accordingly, we assume that there exists a *damage yield function* $\phi(\boldsymbol{\varepsilon},\alpha)$, which is expressed in terms of the strain in order to be able to account for softening behaviors, such that the evolution of α is governed by the *Kuhn-Tucker conditions*:

$$\dot\alpha \geq 0, \qquad \phi(\boldsymbol{\varepsilon},\alpha) \leq 0, \quad \dot\alpha\phi(\boldsymbol{\varepsilon},\alpha) = 0. \tag{6}$$

The first condition in (6) accounts for the *irreversibility*, the second one is the *damage yield criterion* and the third one is the *consistency condition* which expresses that damage can grow only when the strain state is on the yield surface.

One can note that α is the unique internal variable of the model and that is plays also the role of a hardening parameter. The function ϕ is assumed to be sufficiently smooth so that $\phi(\boldsymbol{\varepsilon},\alpha) \leq 0$ corresponds, for every $\alpha\in[0,\alpha_m)$, to a closed connected set in $\mathbb{M}^n_s$ which contains the unstrained state $\boldsymbol{\varepsilon} = 0$, *i.e.* such that ($\phi(0, \alpha) < 0$), and the boundary of which is smooth and evolves continuously with α. This set, denoted $\mathcal{R}(\alpha)$, corresponds to the *elastic domain* (called also the reversibility domain) in the strain space when the material point is in the damaged state α,

$$\mathcal{R}(\alpha) = \{\boldsymbol{\varepsilon} \in \mathbb{M}^n_s \ : \ \phi(\boldsymbol{\varepsilon}, \alpha) \leq 0\}.$$

By virtue of the stress-strain relation (5), when $\alpha < \alpha_m$, one can associate to $\mathcal{R}(\alpha)$ the elastic domain $\mathcal{R}^*(\alpha)$ in the stress space,

$$\mathcal{R}^*(\alpha) = \{\boldsymbol{\sigma} \in \mathbb{M}^n_s \ : \ \phi(\mathsf{S}(\alpha)\boldsymbol{\sigma}, \alpha) \leq 0\}.$$

1.2. Justification of standard models

A priori, the functions ψ et ϕ can be chosen independently. The so-called *standard laws* (Nguyen, 2000) consist in defining the damage yield function ϕ from the thermodynamical force $\mathsf{Y} := -\partial\psi/\partial\alpha$ associated with α. In the present context Y corresponds to an elastic energy release rate. It turns out that the standard law property can be deduced from Drucker-Ilyushin postulate. This fundamental result, which is the cornerstone of all the variational approach developed in the present Lectures, is proved in Marigo (1989) or Marigo (2000) and we merely recall its statement here.

Let α_0 be the initial damage state and let $t \mapsto \boldsymbol{\varepsilon}(t)$ be a cycle in the strain space, *i.e.* a path $\mathbb{M}^n_s$ parameterized by $t \in [0, 1]$ such that $\boldsymbol{\varepsilon}(0) = \boldsymbol{\varepsilon}(1)$. During this cycle imposed to the material point the damage state evolves, its evolution $t \mapsto \alpha(t)$ being governed by the damage law (6). The strain work W done during this cycle is given by

$$W = \int_0^1 \frac{\partial\psi}{\partial\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}(t), \alpha(t)) \cdot \dot{\boldsymbol{\varepsilon}}(t) dt. \tag{7}$$

Drucker-Ilyushin postulate consists in requiring that $W \geq 0$ whatever the initial state α_0 and whatever the cycle which are considered. That leads to the following

Proposition 1.1 (Damage Standard Laws and Drucker-Ilyushin postulate)**.** *The strain work W is non negative for every initial damage state and every strain cycle* only if *the damage criterion is a criterion corresponding to a critical elastic energy release rate criterion. Specifically, there necessarily exists $\kappa(\alpha) > 0$ such that $\mathcal{R}(\alpha)$ can read as*

$$\mathcal{R}(\alpha) = \left\{\boldsymbol{\varepsilon} \in \mathbb{M}^n_s \ : \ -\frac{\partial\psi}{\partial\alpha}(\boldsymbol{\varepsilon}, \alpha) \leq \kappa(\alpha)\right\}. \tag{8}$$

In other words, the yield function ϕ can read as:

$$\phi(\boldsymbol{\varepsilon}, \alpha) = -\frac{\partial\psi}{\partial\alpha}(\boldsymbol{\varepsilon}, \alpha) - \kappa(\alpha). \tag{9}$$

Let us note that : (i) the result is valid even if ψ is not quadratic in $\boldsymbol{\varepsilon}$; (ii) one gets a weaker result when the damage variable is not a scalar, cf Marigo (1989)[Theorems 6.5 and 6.6]; (iii) the converse property is true (and hence Drucker-Ilyushin postulate and critical energy release rate criterion are equivalent) provided additional conditions are assumed for the evolution of $\mathcal{R}(\alpha)$ with α.

The first consequence is that *the strain work is a state function*, *i.e.* the work done in order that the state of the material point evolves from its unstrained and undamaged state $(0,0)$ to the state $(\boldsymbol{\varepsilon}, \alpha)$ is independent of the strain path. Specifically, one gets

$$W = \mathsf{W}_0(\boldsymbol{\varepsilon}, \alpha) := \psi(\boldsymbol{\varepsilon}, \alpha) + w(\alpha) \tag{10}$$

where $\alpha \mapsto w(\alpha)$ is the primitive of $\alpha \mapsto \kappa(\alpha)$ vanishing at $\alpha = 0$, *i.e.*

$$w(\alpha) = \int_0^\alpha \kappa(\beta) d\beta.$$

Accordingly, w corresponds to the energy which is dissipated during the damage process where the damage grows from 0 to α. Since $w' = \kappa > 0$, the dissipated energy is an increasing function of α and hence Clausius-Duhem inequality is automatically satisfied. Indeed, if one considers that the free energy is given by the elastic energy ψ, then the dissipated power $\mathcal{D}$ reads as

$$\mathcal{D} := \boldsymbol{\sigma} \cdot \dot{\boldsymbol{\varepsilon}} - \dot{\psi} = -\frac{\partial \psi}{\partial \alpha}(\boldsymbol{\varepsilon}, \alpha)\dot{\alpha}.$$

Using the consistency equation, one gets

$$\mathcal{D} = w'(\alpha)\dot{\alpha} \geq 0.$$

Finally, the damage evolution law can read (as long as $\alpha < \alpha_m$) as:

$$\boxed{\dot{\alpha}\neg \geq 0, \quad \frac{\partial \mathsf{W}_0}{\partial \alpha}(\boldsymbol{\varepsilon}, \alpha) \geq 0, \quad \dot{\alpha}\frac{\partial \mathsf{W}_0}{\partial \alpha}(\boldsymbol{\varepsilon}, \alpha) = 0}, \tag{11}$$

while the stress-strain relation can also read as

$$\boldsymbol{\sigma} = \frac{\partial \mathsf{W}_0}{\partial \boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}, \alpha).$$

This remark is crucial for obtaining variational properties.

Remark 1. *If one makes the change of variable $\alpha \mapsto \omega = w(\alpha)$, which consists in taking the dissipated energy as the damage variable, the strain work function and the damage criterion can read as*

$$\tilde{\mathsf{W}}_0(\boldsymbol{\varepsilon}, \omega) = \tilde{\psi}(\boldsymbol{\varepsilon}, \omega) + \omega, \qquad -\frac{\partial \tilde{\psi}}{\partial \omega}(\boldsymbol{\varepsilon}, \omega) \leq 1. \tag{12}$$

In the present setting where the elastic energy is a quadratic function of the strain, that leads to

$$\tilde{\mathsf{W}}_0(\boldsymbol{\varepsilon}, \omega) = \tfrac{1}{2}\tilde{\mathsf{E}}(\omega)\boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon} + \omega, \quad -\tfrac{1}{2}\tilde{\mathsf{E}}'(\omega)\boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon} \leq 1,$$

where $\tilde{\mathsf{E}} \circ w = \mathsf{E}$.

1.3. Hardening and softening properties

In the standard model (8), the elastic domains in the strain and the stress spaces read respectively, when $\alpha \neq \alpha_m$:

$$\mathcal{R}(\alpha) = \left\{ \boldsymbol{\varepsilon} \in \mathbb{M}^n_s \ : \ \frac{\partial \mathsf{W}_0}{\partial \alpha}(\boldsymbol{\varepsilon}, \alpha) \geq 0 \right\}, \qquad \mathcal{R}^*(\alpha) = \left\{ \boldsymbol{\sigma} \in \mathbb{M}^n_s \ : \ \frac{\partial \mathsf{W}^*_0}{\partial \alpha}(\boldsymbol{\sigma}, \alpha) \leq 0 \right\} \tag{13}$$

where $\mathsf{W}^*_0(\boldsymbol{\sigma}, \alpha) = \psi^*(\boldsymbol{\sigma}, \alpha) - w(\alpha)$ is the Legendre transform of $\mathsf{W}_0(\varepsilon, \alpha)$ with respect to $\boldsymbol{\varepsilon}$, *i.e.*

$$\mathsf{W}^*_0(\boldsymbol{\sigma}, \alpha) = \sup_{\boldsymbol{\varepsilon} \in \mathbb{M}^n_s} \{ \boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon} - \mathsf{W}_0(\boldsymbol{\varepsilon}, \alpha) \}.$$

In the present context where ψ is a quadratic function of $\boldsymbol{\varepsilon}$, one gets

$$\mathsf{W}^*_0(\boldsymbol{\sigma}, \alpha) = \tfrac{1}{2} \mathsf{S}(\alpha) \boldsymbol{\sigma} \cdot \boldsymbol{\sigma} - w(\alpha).$$

In the present setting and by virtue of the assumption (3), the elastic domains are bounded ellipsoids in the strain and stress spaces which are centered at 0. But more than their shape, it is the evolution of their size with α which plays an essential role in the qualitative properties of the damage evolution problem. To this purpose, let us first introduce the following definition:

Definition 1.2 (Hardening properties). *One says that the behavior of the material is with strain-hardening if $\alpha \mapsto \mathcal{R}(\alpha)$ is increasing, with stress-hardening if $\alpha \mapsto \mathcal{R}^*(\alpha)$ is increasing, with stress-softening (or shortly, with softening) if $\alpha \mapsto \mathcal{R}^*(\alpha)$ is decreasing.*

These monotonicity properties must be understood in the sense of the set inclusion. Accordingly, the strain-hardening property means that $\alpha' > \alpha \Rightarrow \mathcal{R}(\alpha') \supset \mathcal{R}(\alpha)$. If one takes the dissipated energy as the damage variable (cf Remark 1), it turns out that these properties of increase or decrease of the elastic domains are equivalent to properties of convexity or concavity of $\tilde{\mathsf{W}}_0$ or $\tilde{\mathsf{W}}^*_0$ by virtue of the following proposition:

Proposition 1.3 (Convexity and Hardening)**.** *The strain-hardening condition is equivalent to the strict convexity of $\tilde{\mathsf{W}}_0$ with respect to ω at given $\boldsymbol{\varepsilon}$. The stress-hardening condition is equivalent to the strict convexity of $\tilde{\mathsf{W}}_0$ with respect to the pair* couple $(\boldsymbol{\varepsilon}, \omega)$*. The stress-softening condition is equivalent to the strict convexity of $\tilde{\mathsf{W}}^*_0$ with respect to ω at given $\boldsymbol{\sigma}$.*

However, it is important to note that these convexity properties are not invariant by change of the damage variable, while the hardening or softening properties are intrinsic and independent of the choice of the damage variable.

1.4. Some examples

Example 1. *In 1D, the most general model that we can consider in the present setting corresponds to*

$$\mathsf{W}_0(\boldsymbol{\varepsilon},\alpha)=\frac{1}{2}E(\alpha)\boldsymbol{\varepsilon}^2+w(\alpha) \tag{14}$$

where $E(\alpha)$ denotes the Young modulus of the material which decreases from E_0 to 0 when α grows from 0 to α_m. Moreover, $\mathcal{R}(\alpha)$ and $\mathcal{R}^*(\alpha)$ are intervals of the form $[-\varepsilon_c(\alpha),+\varepsilon_c(\alpha)]$ and $[-\sigma_\mathsf{c}(\alpha),+\sigma_\mathsf{c}(\alpha)]$.

Example 2. *In anti-plane elasticity, for an isotropic material the model reads as*

$$\mathsf{W}_0(\boldsymbol{\varepsilon},\alpha)=\mu(\alpha)(\varepsilon_{13}^2+\varepsilon_{23}^2)+w(\alpha) \tag{15}$$

where the direction 3 corresponds to the anti-plane direction. In (15)*, $\mu(\alpha)$ denotes the shear modulus which decreases from μ_0 to 0 when α grows from 0 to α_m. Therefore, $\mathcal{R}(\alpha)$ and $\mathcal{R}^*(\alpha)$ are disks in the planes $(\varepsilon_{13},\varepsilon_{23})$ and $(\sigma_{13},\sigma_{23})$ which are centered at the origin.*

Example 3. *In 3D, still for isotropic materials, the most general model that one can consider in the present setting reads as*

$$\mathsf{W}_0(\boldsymbol{\varepsilon},\alpha)=\frac{1}{2}K(\alpha)(Tr\boldsymbol{\varepsilon})^2+\mu(\alpha)\boldsymbol{\varepsilon}^D\cdot\boldsymbol{\varepsilon}^D+w(\alpha) \tag{16}$$

where $K(\alpha)$ and $\mu(\alpha)$ denote the compressibility and the shear moduli respectively. In (16)*, $\boldsymbol{\varepsilon}^D$ denotes the deviatoric part of $\boldsymbol{\varepsilon}$ and $Tr\boldsymbol{\varepsilon}$ its trace. In general, $\mathcal{R}(\alpha)$ and $\mathcal{R}^*(\alpha)$ are bounded ellipsoids.*

In all the examples above, one can represent any type of hardening or softening properties by a relevant choice of the functions of α entering in their definition.

2. The variational properties of standard models

Let us consider a n-dimensional body whose natural reference configuration is the open set Ω of $\mathbb{R}^n$. It is made of one (or several) brittle damage material(s) of the type described in the previous section. (In the case where the body is heterogeneous, the strain work state function W_0, the ultimate damage state α_m and all other involved quantities depend on the material point x.) From an initial situation, the body is submitted to a loading which is time-dependent.

Denoting by t the time parameter which is defined so that $t = 0$ be the initial time, the *quasi-static* evolution problem consists in seeking for the displacement field $\mathbf{u}_t$, the damage field α_t and the stress field $\boldsymbol{\sigma}_t$ at each time $t \geq 0$. These fields have to satisfy the equilibrium equations, the boundary conditions, the constitutive equations and the damage evolution law. That leads to the following set of conditions:

$$
\begin{aligned}
\text{Equilibrium} &: \operatorname{div} \boldsymbol{\sigma}_t + \mathbf{f}_t = 0 \quad in \quad \Omega \\
\text{Neumann boundary conditions} &: \boldsymbol{\sigma}_t n = \mathbf{F}_t \quad on \quad \partial_F \Omega \\
\text{Dirichlet boundary conditions} &: \mathbf{u}_t = \mathbf{U}_t \quad on \quad \partial_D \Omega \\
\text{Constitutive relations} &: \boldsymbol{\sigma}_t = \mathsf{E}(\alpha_t)\boldsymbol{\varepsilon}(\mathbf{u}_t) \quad in \quad \Omega \\
\text{Compatibility conditions} &: 2\boldsymbol{\varepsilon}(\mathbf{u}_t) = \nabla \mathbf{u}_t + \nabla \mathbf{u}_t^T \quad in \quad \Omega \\
\text{Kuhn-Tucker conditions} &: \begin{cases} \dot{\alpha}_t \geq 0, \\ -\frac{1}{2}\mathsf{E}'(\alpha_t)\boldsymbol{\varepsilon}(\mathbf{u}_t) \cdot \boldsymbol{\varepsilon}(\mathbf{u}_t) \leq w'(\alpha_t) \\ \dot{\alpha}_t \left(\frac{1}{2}\mathsf{E}'(\alpha_t)\boldsymbol{\varepsilon}(\mathbf{u}_t) \cdot \boldsymbol{\varepsilon}(\mathbf{u}_t) + w'(\alpha_t)\right) = 0 \end{cases} \qquad in \quad \Omega
\end{aligned}
$$

at which one must add the initial condition for the damage field (α_0 given in Ω).

In the relations above, $\mathbf{f}_t$ denotes the body forces prescribed at time t, $\mathbf{F}_t$ corresponds to the surface forces prescribed on the part $\partial_F \Omega$ of the boundary at time t and $\mathbf{U}_t$ represents the prescribed displacement of the complementary part $\partial_D \Omega$ of the boundary at time t. Note that we assume that $\partial \Omega$ is divided into two parts, one corresponding to Dirichlet boundary consitions and the other to Neumann boundary conditions, and that these parts do not depend on time. It is of course possible to consider more general boundary conditions, but we adopt these ones to simplify the presentation. Note also that the problem is implicitly set in the framework of small displacements: the equilibrium equations are written in the reference configuration and the relation between the strains and the displacements is linearized, the strain field being the symmetric part of the gradient of the displacement field.

By virtue of the standard character of the damage evolution law (11), the above evolution problem is equivalent to variational inequations. Before to prove this fundamental property, one must introduce some definitions.

Definition 2.1 (Admissible fields). *We denote by $\mathcal{C}_t$ the set of kinematically admissible displacement fields at time t:*

$$\mathcal{C}_t = \{\mathbf{v} \ : \ \mathbf{v} = \mathbf{U}_t \quad on \quad \partial_D \Omega\}.$$

Thus $\mathcal{C}_t$ is an affine space and $\mathcal{C}^0$ is the associated linear space:

$$\mathcal{C}_0 = \{\mathbf{v} \ : \ \mathbf{v} = \mathbf{0} \quad on \quad \partial_D \Omega\}.$$

We assume that the part $\partial_D \Omega$ of the boundary where the displacements are prescribed is sufficiently "large" so that there does not exist admissible rigid displacements. Specifically, we assume that the unique element $\mathbf{v}$ of $\mathcal{C}_0$ which is such that $\boldsymbol{\varepsilon}(\mathbf{v}) = \mathbf{0}$ in Ω is $\mathbf{v} = \mathbf{0}$.

We denote by $\mathcal{D}_0$ *the set of admissible damage fields:*

$$\mathcal{D}_0 = \{\alpha \,:\, 0 \le \alpha \le \alpha_m \text{ in } \Omega\}.$$

With $\alpha \in \mathcal{D}_0$ *we associate the set* $\mathcal{D}(\alpha)$ *of the damage fields which are* accessible *from* α *by taking account of the irreversibility condition:*

$$\mathcal{D}(\alpha) = \{\beta \,:\, \alpha \le \beta \le \alpha_m \text{ dans } \Omega\}.$$

With a pair $(\mathbf{v}, \beta) \in \mathcal{C}_t \times \mathcal{D}_0$ admissible at time t we associate the total energy $\mathcal{E}_t(\mathbf{v}, \beta)$ of the body in this state at that time:

$$\mathcal{E}_t(\mathbf{v}, \beta) = \int_\Omega \mathsf{W}_0(\boldsymbol{\varepsilon}(\mathbf{v}), \beta)dx - W^e_t(\mathbf{v}) \tag{17}$$

where $\boldsymbol{\varepsilon}(\mathbf{v})$ denotes the symmetric part of the gradient of $\mathbf{v}$ (*i.e.* the strain field associated with $\mathbf{v}$) and $W^e_t(\mathbf{v})$ is the work done by the external forces at time t, *i.e.*

$$W^e_t(\mathbf{v}) = \int_\Omega \mathbf{f}_t \cdot \mathbf{v}\ dx + \int_{\partial_F \Omega} \mathbf{F}_t \cdot \mathbf{v}\, dS.$$

Remark 2. *The choice of the functional spaces is a real issue. If* $\beta < \alpha_m$ *almost everywhere, then the energy is finite provided that* $v \in H^1(\Omega, \mathbb{R}^n)$*. But the questions of regularity become much more delicate when a part of the body is fully damaged (i.e. when* $\alpha_t = \alpha_m$ *is a part of* Ω *with non zero volume). This is outside the scope of the present Lectures.*

Let us assume to simplify the presentation that the body is undamaged at time 0, *i.e.* $\alpha_0 = 0$, and that it is free of any loading at this initial time, *i.e.* $\mathbf{U}_0 = \mathbf{0}$, $\mathbf{f}_0 = \mathbf{0}$ and $\mathbf{F}_0 = \mathbf{0}$. Then the body is its natural reference configuration at $t = 0$, $\mathbf{u}_0 = \mathbf{0}$ and $\boldsymbol{\sigma}_0 = \mathbf{0}$. Then, one gets the following fundamental property:

PB 1. *The evolution problem is equivalent to find, for* $t > 0$*,* $(\mathbf{u}_t, \alpha_t) \in \mathcal{C}_t \times \mathcal{D}_0$ *satisfying the following three items*

$$\begin{aligned}
\mathsf{(ir)} &: \dot{\alpha}_t \ge 0, \\
\mathsf{(st)} &: \mathcal{E}'_t(\mathbf{u}_t, \alpha_t)(\mathbf{v} - \mathbf{u}_t, \beta - \alpha_t) \ge 0, \quad \forall (\mathbf{v}, \beta) \in \mathcal{C}_t \times \mathcal{D}(\alpha_t), \\
\mathsf{(eb)} &: \mathcal{E}'_t(\mathbf{u}_t, \alpha_t)(\mathbf{0}, \dot{\alpha}_t) = 0,
\end{aligned}$$

In the statement of the proposition above, $\mathcal{E}'_t(\mathbf{u}_t, \alpha_t)(\mathbf{v}, \beta)$ denotes the directional derivative of $\mathcal{E}_t$ at $(\mathbf{u}_t, \alpha_t)$ in the direction $(\mathbf{v}, \beta)$:

$$\mathcal{E}'_t(\mathbf{u}_t, \alpha_t)(\mathbf{v}, \beta) := \frac{d}{dh}\, \mathcal{E}_t(\mathbf{u}_t + h\mathbf{v}, \alpha_t + h\beta)|_{h=0}\,.$$

Accordingly, in the present context, $\mathcal{E}'_t(\mathbf{u}_t, \alpha_t)(\mathbf{v}, \beta)$ reads as

$$\mathcal{E}'_t(\mathbf{u}_t, \alpha_t)(\mathbf{v}, \beta) = \int_\Omega \Big(\boldsymbol{\sigma}_t \cdot \boldsymbol{\varepsilon}(\mathbf{v}) + \Big(\tfrac{1}{2}\mathsf{E}'(\alpha_t)\boldsymbol{\varepsilon}(\mathbf{u}_t) \cdot \boldsymbol{\varepsilon}(\mathbf{u}_t) + w'(\alpha_t)\Big)\beta\Big)\, dx - W^e_t(\mathbf{v}).$$

Let us note that (ir) and (eb) involve the rate fields whereas (st) only involves the state of the body at a given time.

Proof. The complete proof is left as an exercice. It is based on the fact that the item (st) is simply the variational form of the equilibrium and the damage criterion ($\phi(\boldsymbol{\varepsilon}, \alpha) \leq 0$). The item (eb) is then equivalent to the consistency condition ($\dot{\alpha}\phi(\boldsymbol{\varepsilon}, \alpha) = 0$). □

In practice, in numerical computations one solves the *incremental problem*, *i.e.* the problem obtained after time discretization. It consists in finding, for $i \in \mathbb{N}_*$, $(\mathbf{u}_i, \alpha_i) \in \mathcal{C}_i \times \mathcal{D}(\alpha_{i-1})$ such that

$$\mathcal{E}_i'(\mathbf{u}_i, \alpha_i)(\mathbf{v} - \mathbf{u}_i, \beta - \alpha_i)\neg \geq 0, \quad \forall (\mathbf{v}, \beta) \in \mathcal{C}_i \times \mathcal{D}(\alpha_{i-1}). \tag{18}$$

2.2. The main properties of the evolution problem

The properties are completely different according to the behavior of the material is with stress-hardening or stress-softening.

2.2.1. *Case of stress-hardening*

In that case, the total energy of the body enjoys of convexity properties (see Proposition 1.3). Then, using the incremental problem, one can prove the following fundamental

Proposition 2.2. *If the damage law is a standard damage law with stress-hardening in the sense of Definition 1.2, then, at each time step i when the solution exists,* $(\mathbf{u}_i, \alpha_i)$ *minimizes the total energy* $\mathcal{E}_i$ *of the body over* $\mathcal{C}_i \times \mathcal{D}(\alpha_{i-1})$.

In other words, for stress-hardening materials the incremental problem is equivalent to a sequence of global minimization of the energy. Moreover, if one adopts a stronger condition for the hardening (roughly speaking, with a sufficiently growth of the stresses to infinity when α grows to α_m), then one can prove that the incremental problem admits a unique solution. Accordingly, the standard damage models with stress-hardening are similar from the mathematical viewpoint to standard elasto-plastic materials with hardening.

2.2.2. *Case of stress-softening*

In practice, the materials have a stress-softening behavior before their failure. Conversely, only models with a stress-softening behavior can account for the nucleation of cracks as we will see in the last Lectures. But it is possible to show that any local brittle damage model with stress-softening suffers from the following bad properties:

1. The body cannot sustain too high forces (that corresponds to the concept of limit loads like in plasticity). It is the consequence of the fact that the stresses are bounded. Accordingly, it is possible that the evolution problem admis no solution beyond a certain critical time.
2. On the other hand, even when a solution exists, its uniqueness is not guaranteed. It is even possible to see on very simple examples (like the problem of a bar under a controlled traction at one end) that the evolution problem admits an infinite number of solutions.
3. It can also happen that the evolution problem does not admit a continuous solution in time and that one has to consider discontinuous time evolution of the damage field. This can be seen also in very simple examples where the global response contains a *snap-back*. In such a case, the items (ir) and (eb) have no more sense, at least under the form that they have been formulated.
4. When one tries to find a numerical solution by solving the evolution problem with the finite element method, it turns out that the result is in general very mesh sensitive.

Some of these pathologies are inherent to the stress-softening property while other ones are due to the local character of the model. For instance, it is clear that the concept of limit load is due to the fact that the stresses are bounded and not to the local character of the model. As far as the uniqueness and the mesh sensitivity are concerned, it is necessary to see if some solutions are better than the other ones before to change the model. That requires to introduce a criterion for selecting the solutions. The natural way is to change the item (st) by introducing the concept of *stability*. In the same spirit, it is necessary to change the items (ir) and (eb) to enlarge the set where one searches the solution by allowing time discontinuous solutions.

2.3. The enriched formulation of the evolution problem

2.3.1. *The concept of stability of states*

For conservative systems, one can define the stable states as those which correspond to *local* minima of the energy. This concept can be extended to the dissipated systems which are governed by standard laws, see (Nguyen, 2000). We will follow this way here and we introduce the concept of *directional stability*. That leads to the following definition:

Definition 2.3 (Directional stability). *At a given time t a state $(\mathbf{u},\alpha)$ of the body is said* stable *if it is admissible and if, in any accessible direction, there exists a neighborhood where every accessible direction has an energy which is no less than the energy of the state $(\mathbf{u},\alpha)$.*

Specifically, one requires that $(\mathbf{u},\alpha)\in\mathcal{C}_t\times\mathcal{D}_0$ be such that

$$
(\mathsf{ST})\begin{cases}\forall(\mathbf{v},\beta)\in\mathcal{C}_t\times\mathcal{D}(\alpha),\quad \exists\bar{h}>0,\quad \forall h\in[0,\bar{h}],\\[2ex] \mathcal{E}_t(\mathbf{u}+h(\mathbf{v}-\mathbf{u}),\alpha+h(\beta-\alpha))\geq\mathcal{E}_t(\mathbf{u},\alpha).\end{cases}
$$

Let us remark that if $(\mathbf{u}_t, \alpha_t)$ is stable, then dividing the energy inequality in (ST) by $h > 0$ and passing to the limit when $h \to 0$, one recovers (st). In other words, one has the implication

$$(\mathsf{ST}) \quad \Longrightarrow \quad (\mathsf{st}).$$

So, the equilibrium and the damage yield criterion can be seen as necessary conditions of stability. One says that there are *first order* stability conditions (because they are obtained by expanding the energy of the perturbed system up to the first order in h). But they are not always sufficient to ensure the stability. In other words, (st) and (ST) are not always equivalent. More precisely, (st) and (ST) are equivalent for stress-hardening behaviors but not for stress-softening behaviors. Accordingly, (ST) becomes an interesting criterion of selection in the case of stress-softening where the uniqueness in general fails. We will develop this idea in the next Lectures.

2.3.2. *The energy balance*

The condition (eb) can be seen as a local balance of the total energy when the damage evolves, since it says that the real elastic energy release rate is equal to the dissipated power:

$$-\frac{\partial \psi}{\partial \alpha}(\boldsymbol{\varepsilon}_t, \alpha_t)\dot{\alpha}_t = w'(\alpha_t)\dot{\alpha}_t.$$

To read it under this form supposes that $t \mapsto \alpha_t$ is at least absolutely continuous. In such a case, we have the following

Proposition 2.4 (Global energy balance)**.** *During a smooth damage evolution, the evolution of the total energy satisfies the following global balance*

$$(\mathsf{EB}) \qquad \mathcal{E}_t(\mathbf{u}_t, \alpha_t) = \mathcal{E}_0(\mathbf{u}_0, \alpha_0) + \int_0^t \left(\int_\Omega \boldsymbol{\sigma}_{t'} \cdot \boldsymbol{\varepsilon}(\dot{\mathbf{U}}_{t'})dx - W^e_{t'}(\dot{\mathbf{U}}_{t'}) - \dot{W}^e_{t'}(\mathbf{u}_{t'}) \right) dt'$$

where $\dot{\mathbf{U}}_{t'}$ and $\dot{W}^e_{t'}$ denote the rate of the prescribed loading at time t'.

Formal proof. Differentiating with respect to t the total energy $\mathcal{E}_t(\mathbf{u}_t, \alpha_t)$, taking into account (eb) and using the equilibrium, one gets vient

$$\frac{d}{dt}\mathcal{E}_t(\mathbf{u}_t, \alpha_t) = \int_\Omega \boldsymbol{\sigma}_t \cdot \boldsymbol{\varepsilon}(\dot{\mathbf{u}}_t)dx - W^e_t(\dot{\mathbf{u}}_t) - \dot{W^e}_t(\mathbf{u}_t) = \int_\Omega \boldsymbol{\sigma}_t \cdot \boldsymbol{\varepsilon}(\dot{\mathbf{U}}_t)dx - W^e_t(\dot{\mathbf{U}}_t) - \dot{W}^e_t(\mathbf{u}_t).$$

Integrating with respect to t leads to(EB). □

Let us note that (st) are (eb) equivalent to (st) and (EB) if the evolution is smooth in time, but the advantage of (EB) is that it can be used also when the damage evolution is no more continuous. Indeed, (EB) only involves the regularity of the loading with respect t, *i.e.* the regularity of $t \mapsto \mathbf{U}_t$ and $t \mapsto W^e_t$, and not the regularity of the response. If we adopt (EB) we implicitly require that the total energy be an absolutely continuous function of t even when

$t \mapsto (\mathbf{u}_t, \alpha_t)$ is discontinuous. It is really a strong physical assumption, since we could consider that any non regular time evolution cannot be treated in the framework of quasi-static evolution but involves inertial effects where the kinetic energy plays a role.

2.3.3. *The extended formulation*

Finally, we propose to replace the item (st) by the stability condition (ST) and the item (eb) by its extended version (EB). The new evolution problem reads then:

PB 2 (The extended evolution problem)**.** *Find, for every $t \geq 0$, $(\mathbf{u}_t, \alpha_t) \in \mathcal{C}_t \times \mathcal{D}_0$ such that*

$$\begin{cases} (\mathsf{IR}) : t \mapsto \alpha_t \text{ must be non decreasing;} \\ (\mathsf{ST}) : (\mathbf{u}_t, \alpha_t) \text{ must be stable in the sense of Definition 2.3;} \\ (\mathsf{EB}) : \text{The energy balance must be satisfied.} \end{cases}$$

One sees that in one hand the new formulation is more restrictive than the initial one because one only admits stable states, but is more tolerant in the other hand since one admits non smooth evolutions. Let us emphasize the difference between stability and regularity (or equivalently between instability and non regularity). The concept of stability as it is defined here is a property of a *state* while the regularity is a property of the *evolution*. Accordingly, one can find regular evolutions which satisfy (st) but not (ST) at some times, and, conversely, one can find discontinuous evolutions which satisfy (ST) at each time.

2.4. The necessity for enhancing the local model in the case of softening

Let us use the concept of stability as a criterion of selection of solutions on an example.

Example 4. *Let us consider a one-dimensional bar where both ends are under controlled displacements. According to the behavior is with stress-hardening or stress-softening, we have the following properties:*

- *In the case of a stress-hardening behavior, the evolution problems* **PB 1** *and* **PB 2** *admit the same and unique solution which corresponds to a homogeneous response (the damage field is constant along the bar);*
- *In the case of a stress-softening behavior, the evolution problem* **PB 1** *admits an infinite number of solutions while the evolution problem* **PB 2** *admits no solution beyond the elastic stage. That means that the stability condition* (ST) *is not satisfied by any solution of 1.*

This example suggests that the stability condition is really a good criterion for selecting the solutions, but also that the *local* damage models must be enhanced.

3. The introduction of gradient damage terms

The idea consists in introducing gradient damage terms in the strain work function so that to penalize the localization of damage. That will be made in the restricted setting of linearized theory for isotropic material.

3.1. The enhanced form of the strain work state function

At a given material point, the gradient damage vector $\nabla\alpha$ is now considered as a (local) internal variable as well as the value α of the damage at that point. Accordingly, the state of the material point is characterized by the triplet $(\boldsymbol{\varepsilon}, \alpha, \nabla\alpha)$.

The strain work becomes the following state function

$$\mathsf{W} : \mathbb{M}^n_s \times [0, \alpha_m] \times \mathbb{R}^n \to \mathbb{R}, \quad (\boldsymbol{\varepsilon}, \alpha, \nabla\alpha) \mapsto \mathsf{W}(\boldsymbol{\varepsilon}, \alpha, \nabla\alpha).$$

In the framework of a linearized theory, W is expanded in the neighborhood of the "thermodynamical equilibrium state" $(\mathbf{0}, \alpha, \mathbf{0})$ up to the second order in $\boldsymbol{\varepsilon}$ and $\nabla\alpha$. This linearization is partial in the sense that it only concerns the strain and the gradient of damage but not the damage itself. It is due to the fact that the variation of the stiffness is not small when the damage evolves in the full range $[0, \alpha_m]$. That leads to the following expression

$$\begin{aligned}\mathsf{W}(\boldsymbol{\varepsilon}, \alpha, \nabla\alpha) \;&=\; w(\alpha) + \boldsymbol{\sigma}_0(\alpha) \cdot \boldsymbol{\varepsilon} + \boldsymbol{\tau}(\alpha) \cdot \nabla\alpha \\ &+\; \tfrac{1}{2}\mathsf{E}(\alpha)\boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon} + \boldsymbol{\Lambda}(\alpha) \cdot (\boldsymbol{\varepsilon} \otimes \nabla\alpha) + \tfrac{1}{2}\boldsymbol{\Gamma}(\alpha)\nabla\alpha \cdot \nabla\alpha \end{aligned} \tag{19}$$

where the dot denotes the inner product between vectors or tensors of the same order. In (19) appear new functions of the damage variable, namely $\boldsymbol{\tau}(\alpha) \in \mathbb{R}^n$, $\boldsymbol{\Lambda}(\alpha) \in \mathbb{M}^n_s \otimes \mathbb{R}^n$ and $\boldsymbol{\Gamma}(\alpha) \in \mathbb{M}^n_s$, while $\boldsymbol{\sigma}_0(\alpha)$ denotes a damage dependent prestress. Since we have omitted this term in the local form W_0 of the strain work, we still omit it here:

$$\boldsymbol{\sigma}_0(\alpha) = 0.$$

The other two terms $w(\alpha)$ and $\frac{1}{2}\mathsf{E}(\alpha)\boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon}$ were already in W_0.

If we assume that the material is isotropic and that the damage variable is a an objective scalar (*i.e.* invariant in any change of frame), then W must satisfy the following invariance conditions:

$$\mathsf{W}(Q\boldsymbol{\varepsilon}Q^T, \alpha, Q\nabla\alpha) = \mathsf{W}(\boldsymbol{\varepsilon}, \alpha, \nabla\alpha), \qquad \forall Q \in \mathbb{O}^n, \quad \forall(\boldsymbol{\varepsilon}, \alpha, \nabla\alpha) \in \mathbb{M}^n_s \times [0, \alpha_m] \times \mathbb{R}^n \tag{20}$$

where $\mathbb{O}^n$ denotes the complete orthogonal group. Consequently, it is possible to prove that the vector $\boldsymbol{\tau}(\alpha)$ and the third order tensor $\boldsymbol{\Lambda}(\alpha)$ necessarily vanish:

$$\boldsymbol{\tau}(\alpha) = \mathbf{0}, \qquad \boldsymbol{\Lambda}(\alpha) = \mathbf{0}.$$

Moreover the second order tensor $\mathbf{\Gamma}(\alpha)$ is necessarily proportional to the identity. For obvious reasons, if one wants that the gradient damage terms have regularizing effects, then $\mathbf{\Gamma}(\alpha)$ must be positive definite. That allows us to write

$$\mathbf{\Gamma}(\alpha) = \gamma(\alpha)\mathbf{I} \quad \text{with} \quad \gamma(\alpha) > 0.$$

To summarize, for an isotropic material, in a linearized theory and in the absence of prestress, the strain work state function can read as

$$\mathsf{W}(\boldsymbol{\varepsilon}, \alpha, \nabla\alpha) = w(\alpha) + \tfrac{1}{2}\mathsf{E}(\alpha)\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon} + \tfrac{1}{2}\gamma(\alpha)\nabla\alpha\cdot\nabla\alpha. \tag{21}$$

One sees that W differs from its local homologous W_0 only by the quadratic term in $\nabla\alpha$ which involves a new positive scalar function of the damage variable.

3.2. Normalization of the damage variable

At this stage the choice of the damage variable is arbitrary and has no other constraint but that to describe the evolution of the mechanical properties of the material. The damage parameter can be changed via a change of variable without changing the model. An infinite number of choices are possible. A particularly interesting choice consists in taking for the damage variable the volumic dissipated energy in a homogeneous damage process. That consists in taking $\omega = w(\alpha)$ as the damage variable. With this choice, the hardening properties are simply expressed in terms of the convexity of the strain work function, see Proposition 1.3.

Another interesting choice from a practical viewpoint consists in changing the damage variable so that the mutiplicative factor $\gamma(\alpha)$ becomes a constant. It is even possible to take this constant equal to 1. Indeed, making the change of variable

$$\alpha \mapsto D = \Delta(\alpha) := \int_0^\alpha \sqrt{\gamma(\beta)}d\beta,$$

the new strain work function reads as

$$\hat{W}(\boldsymbol{\varepsilon}, D, \nabla D) = w\circ\Delta^{-1}(D) + \tfrac{1}{2}\mathsf{E}\circ\Delta^{-1}(D)\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon} + \tfrac{1}{2}\nabla D\cdot\nabla D.$$

However, the variable D has a physical dimension and varies in a range which depends on the material. In order to compare different materials, it can be interesting to normalize the damage variable so that it becomes dimensionless and varies in a fixed interval. For instance, in the case where $\Delta(\alpha_m) < +\infty$, making the change of variable $\alpha \mapsto d = \Delta(\alpha)/\Delta(\alpha_m)$, the damage variable is dimensionless and runs in the interval $[0, 1]$. However the strain work function becomes

$$\check{\mathsf{W}}(\boldsymbol{\varepsilon}, d, \nabla d) = \check{w}(d) + \frac{1}{2}\check{\mathsf{E}}(d)\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon} + \frac{\gamma}{2}\nabla d\cdot\nabla d \tag{22}$$

and $\gamma > 0$ is now a positive *material constant* which has the dimension of a force.

In turn, the constant γ can be also normalized, for instance by setting $\gamma = E_0\ell^2$, E_0 denoting the Young modulus of the undamaged material. In such a case, $\ell > 0$ is a *material characteristic length*. But, if one changes the normalization, for instance by setting $\gamma = \breve{w}(1)\breve{\ell}^2$, then one changes the definition of the characteristic length. So, the characteristic length also depends on the choice of the normalization. With the choice of d as the damage variable, the behavior of the material is finally characterized by the functions $\breve{w}$, $\breve{\mathsf{E}}$ and the constant γ.

Conclusion: In the remaining part of these lectures, we will assume that the damage variable has been chosen in such a manner that it runs in the interval $[0,1]$ *and that* γ *is a constant with respect to the damage variable (but* γ *can depend on the material point). This damage variable will be still denoted by* α*. Accordingly, the strain work function reads as:*

$$\mathsf{W}(\boldsymbol{\varepsilon}, \alpha, \nabla\alpha) = \mathsf{W}_0(\boldsymbol{\varepsilon}, \alpha) + \frac{\gamma}{2}\nabla\alpha \cdot \nabla\alpha \tag{23}$$

with

$$\mathsf{W}_0(\boldsymbol{\varepsilon}, \alpha) = w(\alpha) + \frac{1}{2}\mathsf{E}(\alpha)\boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon}. \tag{24}$$

4. The variational formulation of the evolution problem

4.1. The three physical principles

Because of the non local character of the model, the damage evolution law must be formulated at the level of the whole structure. For that, we propose to use the same three principles of irreversibility, stability and energy balance as those introduced for the local models. The unique change is that now the total energy of the structure involves the strain work function W instead of W_0.

Accordingly, with $(\mathbf{v}, \beta) \in \mathcal{C}_t \times \mathcal{D}_0$, *i.e.* with a pair admissible at time t, is associated the total energy $\mathcal{E}_t(\mathbf{v}, \beta)$ of the body in its state

$$\mathcal{E}_t(\mathbf{v}, \beta) = \int_\Omega \mathsf{W}(\boldsymbol{\varepsilon}(\mathbf{v}), \beta, \nabla\beta)dx - W^e_t(\mathbf{v}). \tag{25}$$

An important change is that the gradient damage field must be in the space $L^2(\Omega)$ of functions which are square integrable in order that the energy of the body be finite. The consequence is that the trace of the damage fields can be defined on the boundary of the domain. Hence one can prescribe the value of the damage on $\partial\Omega$. However, to simplify the presentation, we will assume that no constraint is prescribed to the damage on the boundary. Accordingly, the set of admissible damage fields is $\mathcal{D}_0 = H^1(\Omega, [0,1])$.

The evolution problem reads as

PB 3. *The evolution problem consists in finding, for every $t \geq 0$, $(\mathbf{u}_t, \alpha_t) \in \mathcal{C}_t \times \mathcal{D}_0$ such that*

(IR) $t \mapsto \alpha_t$ *must be non decreasing;*

(ST) $(\mathbf{u}_t, \alpha_t)$ *must be stable in the sense that:*

$$\forall (\mathbf{v}, \beta) \in \mathcal{C}_t \times \mathcal{D}(\alpha_t), \quad \exists \bar{h} > 0, \quad \forall h \in [0, \bar{h}],$$

$$\mathcal{E}_t(\mathbf{u}_t + h(\mathbf{v} - \mathbf{u}_t), \alpha_t + h(\beta - \alpha_t)) \geq \mathcal{E}_t(\mathbf{u}_t, \alpha_t);$$

(EB) *The balance of energy must hold:*

$$\mathcal{E}_t(\mathbf{u}_t, \alpha_t) = \mathcal{E}_0(\mathbf{u}_0, \alpha_0) + \int_0^t \left(\int_\Omega \boldsymbol{\sigma}_{t'} \cdot \boldsymbol{\varepsilon}(\dot{\mathbf{U}}_{t'}) dx - W^e_{t'}(\dot{\mathbf{U}}_{t'}) - \dot{W}^e_{t'}(\mathbf{u}_{t'}) \right) dt'.$$

The evolution problem **PB 3** is formally the same as **PB 2**. But, of course, because of the presence of gradient damage terms in the energy, the two problems do not admit the same solutions. Note also that the properties of irreversibility, stability and energy balance are invariant by change of the damage variable. Therefore, if they are satisfied for the normalized model, they are satisfied by any other represent of the same model.

4.2. The first order local conditions

Let us assume that **PB 3** admits a solution, that this solution is regular both in space and time. Moreover, we will only consider times when there is no part in the body which is fully damaged, *i.e.* we assume that $\alpha_t < 1$ everywhere in Ω. Let us derive necessary conditions that such a solution must satisfy. Dividing (ST) by $h > 0$ and passing to the limit when $h \to 0$, one obtains the first order conditions that $(\mathbf{u}_t, \alpha_t)$ must satisfy at time t:

$$\mathcal{E}'_t(\mathbf{u}_t, \alpha_t)(\mathbf{v} - \mathbf{u}_t, \beta - \alpha_t) \geq 0, \quad \forall (\mathbf{v}, \beta) \in \mathcal{C}_t \times \mathcal{D}(\alpha_t) \tag{26}$$

where $\mathcal{E}'_t(\mathbf{u}_t, \alpha_t)(\bar{\mathbf{u}}, \bar{\beta})$ denotes the directional derivative of $\mathcal{E}_t$ at $(\mathbf{u}_t, \alpha_t)$ in the direction $(\bar{\mathbf{u}}, \bar{\beta})$, *i.e.* the linear form defined by

$$\mathcal{E}'_t(\mathbf{u}_t, \alpha_t)(\bar{\mathbf{u}}, \bar{\beta}) = \int_\Omega \boldsymbol{\sigma}_t \cdot \boldsymbol{\varepsilon}(\bar{\mathbf{u}}) dx - W^e_t(\bar{\mathbf{u}}) + \int_\Omega \left(\frac{\partial \mathsf{W}_0}{\partial \alpha}(\boldsymbol{\varepsilon}(\mathbf{u}_t), \alpha_t)\bar{\beta} + \gamma \nabla \alpha_t \cdot \nabla \bar{\beta} \right) dx.$$

Taking $\beta = \alpha_t$ in (26) and noting that $\mathcal{C}_t$ is an affine space, then one recovers the variational formulation of the equilibrium, *i.e.*

$$\int_\Omega \boldsymbol{\sigma}_t \cdot \boldsymbol{\varepsilon}(\bar{\mathbf{u}}) dx = W^e_t(\bar{\mathbf{u}}), \quad \forall \bar{\mathbf{u}} \in \mathcal{C}^0. \tag{27}$$

Inserting into (26) gives the variational formulation of the non local damage criterion. Specifically, one gets

$$\int_\Omega \Big(\frac{\partial \mathsf{W}_0}{\partial \alpha}(\boldsymbol{\varepsilon}(\mathbf{u}_t),\alpha_t)(\beta-\alpha_t)+\gamma\nabla\alpha_t\cdot\nabla(\beta-\alpha_t)\Big)dx \geq 0, \quad \forall \beta \in \mathcal{D}(\alpha_t). \tag{28}$$

After an integration by parts in (28), one gets: $\forall \beta \in \mathcal{D}(\alpha_t)$,

$$\int_\Omega \Big(\frac{\partial \mathsf{W}_0}{\partial \alpha}(\boldsymbol{\varepsilon}(\mathbf{u}_t),\alpha_t) - \mathrm{div}\left(\gamma\, \mathrm{grad}\, \alpha_t\right)\Big)(\beta-\alpha_t)dx + \int_{\partial\Omega}\gamma\frac{\partial \alpha_t}{\partial n}(\beta-\alpha_t)dS \geq 0 \tag{29}$$

where n denotes the unit outer normal to Ω. Under the condition that the fields are sufficiently smooth, a classical argument of Calculus of Variations leads to the following local conditions

$$\boxed{\frac{\partial \mathsf{W}_0}{\partial \alpha}(\boldsymbol{\varepsilon}(\mathbf{u}_t),\alpha_t) - \mathrm{div}\left(\gamma\, \mathrm{grad}\, \alpha_t\right) \geq 0 \quad \text{in} \quad \Omega}, \qquad \boxed{\gamma\frac{\partial \alpha_t}{\partial n} \geq 0 \quad \text{on} \quad \partial\Omega}. \tag{30}$$

Therefore, (30) constitutes the damage yield criterion for a gradient damage model. If we compare them to their homologous (11) for local damage models, one sees that the damage yield criterion in the bulk now contains a term involving the second spatial derivatives of the damage field. Let us also that they give natural boundary conditions for the normal derivative of the damage field, but that those conditions are inequalities.

Let us now use the energy balance with the same assumptions on the regularity and the absence of fully damaged zone. Differentiating (EB) with respect to t leads to

$$\begin{aligned}
0 &= \frac{d}{dt}\mathcal{E}_t(\mathbf{u}_t,\alpha_t) - \int_\Omega \boldsymbol{\sigma}_t\cdot\boldsymbol{\varepsilon}(\dot{\mathbf{U}}_t)dx + W_t^e(\dot{\mathbf{U}}_t) + \dot{W}_t^e(\mathbf{u}_t)\\
&= \int_\Omega \boldsymbol{\sigma}_t\cdot\boldsymbol{\varepsilon}(\dot{u}_t-\dot{\mathbf{U}}_t)dx - W_t^e(\dot{\mathbf{u}}_t-\dot{\mathbf{U}}_t) + \int_\Omega\Big(\frac{\partial \mathsf{W}_0}{\partial \alpha}(\boldsymbol{\varepsilon}(\mathbf{u}_t),\alpha_t)\dot{\alpha}_t + \gamma\nabla\alpha_t\cdot\nabla\dot{\alpha}_t\Big)dx\\
&= \int_\Omega\Big(\frac{\partial \mathsf{W}_0}{\partial \alpha}(\boldsymbol{\varepsilon}(\mathbf{u}_t),\alpha_t)\dot{\alpha}_t + \gamma\nabla\alpha_t\cdot\nabla\dot{\alpha}_t\Big)dx.
\end{aligned}$$

The first two terms of the second line cancel by virtue of the equilibrium (27). Integrating by parts the damage gradient term leads to

$$0 = \int_\Omega\Big(\frac{\partial \mathsf{W}_0}{\partial \alpha}(\boldsymbol{\varepsilon}(\mathbf{u}_t),\alpha_t) - \mathrm{div}(\gamma\, \mathrm{grad}\, \alpha_t)\Big)\dot{\alpha}_t dx + \int_{\partial\Omega}\gamma\frac{\partial \alpha_t}{\partial n}\dot{\alpha}_t dS.$$

Finally, it suffices to take account of the irreversibility condition which requires that $\dot{\alpha}_t \geq 0$ and of the damage criterion (30) for obtaining the desired consistency conditions

$$\boxed{\dot{\alpha}_t\Big(\frac{\partial \mathsf{W}_0}{\partial \alpha}(\boldsymbol{\varepsilon}(\mathbf{u}_t),\alpha_t) - \mathrm{div}\left(\gamma\, \mathrm{grad}\, \alpha_t\right)\Big) = 0 \quad \text{in} \quad \Omega}, \qquad \boxed{\dot{\alpha}_t\gamma\frac{\partial \alpha_t}{\partial n} = 0 \quad \text{on} \quad \partial\Omega}. \tag{31}$$

Remark 3. *The set* (30)-(31) *of local conditions for gradient damage models are obtained here from the variational principles of stability and energy balance. They correspond to the conditions which are set a priori in the literature, see* Comi *(*1999*),* Lorentz and Andrieux *(*2003*).*

4.3. The second order stability conditions

The damage yield conditions (30) are only necessary stability conditions. They are in general not sufficient and one must also consider second order stability conditions in order that (ST) be satisfied. Let us establish these second order stability conditions. Let $(\mathbf{u}_t, \alpha_t)$ be a state at time t which satisfies the first order stability conditions (26) (or equivalently (27) and (28)). Let us set $(\bar{\mathbf{u}}, \bar{\beta}) = (\mathbf{v} - \mathbf{u}_t, \beta - \alpha_t)$ and let us express (ST) by expanding $\mathcal{E}_t(\mathbf{u}_t + h\bar{\mathbf{u}}, \alpha_t + h\bar{\beta})$ with respect to h up to the second order. That leads to

$$0 \leq h\mathcal{E}_t'(\mathbf{u}_t, \alpha_t)(\bar{\mathbf{u}}, \bar{\beta}) + \frac{h^2}{2}\mathcal{E}_t''(\mathbf{u}_t, \alpha_t)(\bar{\mathbf{u}}, \bar{\beta}) + o(h^2) \tag{32}$$

where $\mathcal{E}_t''(\mathbf{u}_t, \alpha_t)$ denotes the second derivative of $\mathcal{E}_t$ at $(\mathbf{u}_t, \alpha_t)$. It is a quadratic form with respect to $(\bar{\mathbf{u}}, \bar{\beta})$. Using the compliance function $\mathsf{S}(\alpha) = \mathsf{E}(\alpha)^{-1}$ and this derivative with respect to α, the second derivative of the energy $\mathcal{E}_t''(\mathbf{u}_t, \alpha_t)$ reads as

$$\begin{aligned}\mathcal{E}_t''(\mathbf{u}_t, \alpha_t)(\bar{\mathbf{u}}, \bar{\beta}) \ = \ & \int_\Omega \mathsf{E}(\alpha_t)\big(\boldsymbol{\varepsilon}(\bar{\mathbf{u}}) - \bar{\beta}\,\mathsf{S}'(\alpha_t)\boldsymbol{\sigma}_t\big) \cdot \big(\boldsymbol{\varepsilon}(\bar{\mathbf{u}}) - \bar{\beta}\,\mathsf{S}'(\alpha_t)\boldsymbol{\sigma}_t\big)dx + \int_\Omega \gamma\nabla\bar{\beta}\cdot\nabla\bar{\beta}dx \\ & + \int_\Omega \big(w''(\alpha_t) - \tfrac{1}{2}\mathsf{S}''(\alpha_t)\boldsymbol{\sigma}_t \cdot \boldsymbol{\sigma}_t\big)\bar{\beta}^2 dx.\end{aligned} \tag{33}$$

By virtue of (26), the first term in the right hand side of (32) is non negative. If it is positive, then, for h small enough, the inequality (32) is satisfied and the state $(\mathbf{u}_t, \alpha_t)$ is stable in the direction $(\bar{\mathbf{u}}, \bar{\beta})$. On the other hand, if the first order term vanishes, then the state is stable in the direction $(\bar{\mathbf{u}}, \bar{\beta})$ *only if* the second derivative is non negative (and it will be stable in that direction *if* the second derivative is positive).

Let us note that the first two terms in the expression (33) of the second derivative are non negative by virtue of the positivity of E and γ. The sign of the last term in (33) depends on the sign of $w''(\alpha_t) - \frac{1}{2}\mathsf{S}''(\alpha_t)\boldsymbol{\sigma}_t \cdot \boldsymbol{\sigma}_t$ which is positive in the case of a stress-hardening behavior and negative in the case of a stress-softening behavior. Let us discriminate between the two cases.

1. *Case of stress-hardening.* In such a case the second derivative is positive in any direction $(\bar{\mathbf{u}}, \bar{\beta}) \neq (0, 0)$. Indeed, the second derivative is non negative. It can vanish only if $\bar{\beta} = 0$ and $\boldsymbol{\varepsilon}(\bar{u}) = 0$ in Ω and hence, by virtue of the assumption made in Definition 2.1, only if $(\bar{\mathbf{u}}, \bar{\beta}) = (0, 0)$. Therefore, for h small enough, the inequality (32) is satisfied and the state $(\mathbf{u}_t, \alpha_t)$ is stable in any admissible direction. So, we have obtained that, in the case of stress-hardening, any state which satisfies the first order stability condition (st) automatically satisfies the stability condition (ST). In other words, the problems **PB 1** and **PB 2** are equivalent.

2. *Case of stress-softening.* By virtue of (29), the first order term $\mathcal{E}_t'(\mathbf{u}_t, \alpha_t)(\bar{\mathbf{u}}, \bar{\beta})$ vanishes if and only if $\bar{\beta} = 0$ in the part of Ω and the part of $\partial\Omega$ where the equality does not hold in (30). Therefore, let us define Ω_t^a and $\partial_t^a\Omega$ as the part of the domain and the part of the

boundary where the damage criterion is satisfied as an equality (we can call such points *damaging points*). Specifically,

$$\Omega_t^a = \left\{ x \in \Omega_t^d \,:\, \frac{\partial \mathsf{W}_0}{\partial \alpha}(\boldsymbol{\varepsilon}(\mathbf{u}_t), \alpha_t) - \operatorname{div}\left(\gamma \operatorname{grad} \alpha_t\right) = 0 \right\}, \tag{34}$$

$$\partial_t^a \Omega = \left\{ x \in \partial_t^d \Omega \,:\, \frac{\partial \alpha_t}{\partial n} = 0 \right\}. \tag{35}$$

Let us consider the directions $(\bar{\mathbf{u}}, \bar{\beta}) \in \mathcal{C}^0 \times \mathcal{D}_t^a$ with

$$\mathcal{D}_t^a = \{\bar{\beta} \in H^1(\Omega) \,:\, \bar{\beta} \geq 0,\ \bar{\beta} = 0 \text{ in } \Omega \backslash \Omega_t^a,\ \bar{\beta} = 0 \text{ on } \partial\Omega \backslash \partial_t^a \Omega\}.$$

In such a direction (and only in these directions), one has $\mathcal{E}_t'(\mathbf{u}_t, \alpha_t)(\bar{\mathbf{u}}, \bar{\beta}) = 0$. Accordingly, the stability of the state in such a direction will depend on the sign of the second derivative. But in the case of stress-softening, the second derivative is the difference between two non negative quadratic forms (one corresponding to the sum of the first two terms, the other to the third term). Accordingly, the study of the sign of the second derivative is equivalent to compare the ratio of the two quadratic forms to 1. That leads to consider the following *Rayleigh ratio* defined on $\mathcal{C}^0 \times \mathcal{D}_t^a$:

$$\mathcal{R}_t(\bar{\mathbf{u}}, \bar{\beta}) = \frac{\displaystyle\int_\Omega \mathsf{E}(\alpha_t)\big(\boldsymbol{\varepsilon}(\bar{\mathbf{u}}) - \bar{\beta}\, \mathsf{S}'(\alpha_t)\boldsymbol{\sigma}_t\big) \cdot \big(\boldsymbol{\varepsilon}(\bar{\mathbf{u}}) - \bar{\beta}\, \mathsf{S}'(\alpha_t)\boldsymbol{\sigma}_t\big) dx + \int_{\Omega_t^a} \gamma \nabla\bar{\beta} \cdot \nabla\bar{\beta} dx}{\displaystyle\int_{\Omega_t^a} \big(\tfrac{1}{2}\mathsf{S}''(\alpha_t)\boldsymbol{\sigma}_t \cdot \boldsymbol{\sigma}_t - w''(\alpha_t)\big)\bar{\beta}^2 dx} \tag{36}$$

In (36) we implicitly assume that $\mathcal{R}_t(\bar{\mathbf{u}}, 0) = +\infty$.

Therefore, the state $(\mathbf{u}_t, \alpha_t)$ will be stable or unstable according to the minimum of the Rayleigh ratio over $\mathcal{C}^0 \times \mathcal{D}_t^a$ is greater or less than 1. Since $\gamma > 0$, one sees that the gradient damage terms have a stabilizing effect. Let us note also that this second order stability condition is necessarily *global*, it is a *structural property* and not only a *material property.*

Let us summarize all the results that we have obtained in the present section by the following proposition:

Proposition 4.1. *In order that an evolution $t \mapsto (\mathbf{u}_t, \alpha_t)$, which starts from $(\mathbf{u}_0, \alpha_0)$ and is regular both in space and time, satisfies the evolution problem* (IR), (ST) *and* (EB), *it is* necessary *that this evolution satisfies at each time the equilibrium* (27)*, the irreversibility condition, the damage criterion* (30) *and the consistency condition* (31)*. In the case of a stress-hardening behavior, it is* sufficient. *However, in the case of a stress-softening, it is also* necessary *that the minimum of the Rayleigh ratio be greater or equal to 1 at each time.*

Part II

Stability and uniqueness of homogeneous responses

Abstract

A bifurcation and stability analysis is carried out here for a bar made of a material obeying a gradient damage model with softening. We show that the associated initial boundary-value problem is ill-posed and one should expect mesh sensitivity in numerical solutions. However, in contrast to what happens for the underlying local damage model, the damage localization zone has a finite thickness and stability arguments can help in the selection of solutions. The matter of this part is essentially borrowed from Benallal and Marigo (2007). A more complete analysis can also be found in Pham et al. (2011b).

5. Setting of the problem

5.1. Constitutive assumptions

We consider a one-dimensional non local damage model in which the damage variable α is a scalar growing from 0 to infinity[1]. The behavior of the material at a material point x is characterized by the state function W which depends on the local strain $u'(x)$ (u denoting the displacement and the prime denoting the spatial derivative), the local damage value $\alpha(x)$ and the gradient $\alpha'(x)$ of the damage field at x:

$$\mathsf{W}(u', \alpha, \alpha') = \tfrac{1}{2}E(\alpha)u'^2 + w(\alpha) + \tfrac{1}{2}E_0\ell^2\alpha'^2 \tag{37}$$

where $E(\alpha)$ represents the Young modulus of the material at the damage state α and $w(\alpha)$ can be interpreted as the density of the dissipated energy by the material during a homogeneous damaging process (*i.e.* a process such that $\alpha' = 0$). The constant in front of the damage gradient term is normalized and ℓ represents the internal length of the material associated with this normalization.

5.2. Variational approach in the case of the traction of a bar

In the spirit of the first part of these Lectures, we formulate here the damage constitutive equations at the level of the whole structure. We consider a homogeneous bar whose natural reference configuration is the interval $(0, L)$. Thus, L is the bar length. The bar is made of the non local damaging material characterized by the state function W given by (37). The end $x = 0$ of the bar is fixed, while the displacement of the end $x = L$ is prescribed by a hard device to a

[1] Accordingly, in the present model $\alpha_m = +\infty$ and hence the material cannot be fully damaged. This restricted assumption will be removed in the next part so that cracks can nucleate.

value increasing with time from 0 to infinity:

$$u_t(0) = 0, \qquad u_t(L) = t\varepsilon_1 L, \qquad t \geq 0, \tag{38}$$

where t denotes the time, u_t is the displacement field of the bar at time t and $\varepsilon_1 = \dfrac{\sigma_1}{E_0}$ is the elastic yield strain.

The damage evolution problem of the bar is obtained via an energetic variational formulation. We recall briefly the basic ingredients of a such variational formulation. Let $\mathcal{C}_t$ and $\mathcal{D}$ be respectively the affine space of kinematically admissible displacement fields at time t and the convex cone of admissible damage fields

$$\mathcal{C}_t = \{v \in H^1(0, L) \;:\; v(0) = 0, v(L) = t\varepsilon_1 L\}, \tag{39}$$

$$\mathcal{D} = \{\beta \in H^1(0, L) \;:\; \beta(x) \geq 0\}, \tag{40}$$

$H^1(0, L)$ denoting the usual Sobolev space of functions defined on $(0, L)$ which are square integrable and the first derivative of which is also square integrable. The linear space associated to $\mathcal{C}_t$ is $H^1_0(0, L) = \{v \in H^1(0, L) \;: v(0) = v(L) = 0\}$.

At any pair (u, α) admissible at time t, we associate *the total energy of the bar*

$$(u, \alpha) \in \mathcal{C}_t \times \mathcal{D} \mapsto \mathcal{E}(u, \alpha) = \int_0^L \mathsf{W}(u'(x), \alpha(x), \alpha'(x)) \; dx. \tag{41}$$

Note that the energy functional does not depend explicitly on time.

By assuming that the bar is undamaged at time $t = 0$, the damage evolution problem can be read as (see part I):

PB 4. *The evolution problem consists in finding, for $t > 0$, $(u_t, \alpha_t) \in \mathcal{C}_t \times \mathcal{D}$ satisfying the following three items*

$$\begin{aligned} (\mathsf{ir}) &\;:\; \dot{\alpha}_t \geq 0, \\ (\mathsf{st}) &\;:\; \mathcal{E}'(u_t, \alpha_t)(v - u_t, \beta - \alpha_t) \geq 0, \quad \forall (v, \beta) \in \mathcal{C}_t \times \mathcal{D}(\alpha_t), \\ (\mathsf{eb}) &\;:\; \mathcal{E}'(u_t, \alpha_t)(0, \dot{\alpha}_t) = 0, \end{aligned}$$

with the initial condition

$$\alpha_0(x) = 0. \tag{42}$$

In the statement above $\mathcal{E}'(u, \alpha)(v, \beta)$ denotes the Gâteaux derivative of $\mathcal{E}$ at (u, α) in the direction (v, β), *i.e.*

$$\mathcal{E}'(u, \alpha)(v, \beta) = \int_0^L E(\alpha) u' v' dx + \int_0^L \Big(\Big(\tfrac{1}{2} E'(\alpha) u'^2 + w'(\alpha) \Big) \beta + E_0 \ell^2 \alpha' \beta' \Big) \, dx. \tag{43}$$

The set of admissible displacement rates $\dot{u}$ (the dot denoting the time derivative) can be identified with $\mathcal{C}_1$, while the set of admissible damage rates $\dot{\alpha}$ can be identified with $\mathcal{D}$, the inequality $\dot{\alpha} \geq 0$ denoting the irreversibility of the damaging process. Accordingly, the evolution problem can be read in a more condensed form by virtue of the following proposition:

Proposition 5.1. *The evolution problem* **PB 4** *is equivalent to*

$$\begin{aligned} &\text{For } t > 0, \text{ find } (u_t, \alpha_t) \in \mathcal{C}_t \times \mathcal{D} \text{ such that } (\dot{u}_t, \dot{\alpha}_t) \in \mathcal{C}_1 \times \mathcal{D} \\ &\text{and} \quad \mathcal{E}'(u_t, \alpha_t)(v - \dot{u}_t, \beta - \dot{\alpha}_t) \geq 0 \quad \forall (v, \beta) \in \mathcal{C}_1 \times \mathcal{D} \end{aligned} \tag{44}$$

with the initial condition (42)*.*

Proof. The proof is left as an exercice. □

5.3. Nonlinear initial boundary-value problem for the bar

By inserting $\beta = \dot{\alpha}_t$ and $v = \dot{u}_t + w$, with $w \in H^1_0(0, L)$, into(44), we obtain the variational formulation of the equilibrium of the bar, *i.e.*

$$\int_0^L E(\alpha_t(x)) u'(x) w'(x)\, dx = 0 \qquad \forall w \in H^1_0(0, L), \tag{45}$$

from which we deduce that the stress must be uniform:

$$E(\alpha_t(x)) u'_t(x) = \sigma_t, \quad \forall x \in (0, L). \tag{46}$$

By using the boundary condition (38), we obtain the relation between the stress σ_t and the damage field α_t

$$\varepsilon_t \equiv t\varepsilon_1 = \frac{\sigma_t}{L} \int_0^L \frac{dx}{E(\alpha_t(x))}, \tag{47}$$

ε_t representing the overall strain of the bar at time t.

By inserting (45)–(47) into (44) we obtain the variational inequality governing the damage field evolution:

$$\tfrac{1}{2}\sigma_t^2 \int_0^L \frac{E'(\alpha_t)}{E(\alpha_t)^2} \beta dx + \int_0^L w'(\alpha_t) \beta dx + E_0 \ell^2 \int_0^L \alpha'_t \beta' dx \geq 0, \tag{48}$$

where the inequality must hold for all $\beta \in \mathcal{D}$ and becomes an equality when $\beta = \dot{\alpha}_t$. After an integration by parts and by using classical arguments of the calculus of variations, we obtain the set of local conditions satisfied by the damage field at any point of $(0, L)$

$$\text{Irreversibility condition} \; : \; \dot{\alpha}_t \geq 0 \tag{49}$$

$$\text{Damage yield criterion} \; : \; f := \tfrac{1}{2}\sigma_t^2 \frac{E'(\alpha_t)}{E(\alpha_t)^2} + w'(\alpha_t) - E_0\ell^2\alpha_t'' \geq 0 \tag{50}$$

$$\text{Consistency condition} \; : \; \dot{\alpha}_t \left(\tfrac{1}{2}\sigma_t^2 \frac{E'(\alpha_t)}{E(\alpha_t)^2} + w'(\alpha_t) - E_0\ell^2\alpha_t'' \right) = 0 \tag{51}$$

with the boundary conditions

$$\alpha_t'(0) \leq 0, \quad \dot{\alpha}_t(0)\alpha_t'(0) = 0, \qquad \alpha_t'(L) \geq 0, \quad \dot{\alpha}_t(L)\alpha_t'(L) = 0 \tag{52}$$

and the initial condition (42). Conditions (49), (50) and (51) are nothing else than the Kuhn-Tucker conditions $\dot{\alpha}_t \geq 0$, $f \geq 0$ and $f\dot{\alpha}_t = 0$.

5.4. Rate boundary-value problem

The properties of bifurcation, uniqueness or stability of the solutions of the damage evolution problem can be obtained by analyzing the rate damage problem, *i.e.* the problem governing at a given time t the damage rate $\dot{\alpha}_t$ by assuming that the damage state α_t is known. In its variational form, the damage rate problem is obtained by differentiating the damage evolution problem. Let us briefly recall how it is obtained.

Let α_t be an admissible damage field and let u_t be the associated displacement field giving the equilibrium of the bar at time t, see (46). The total energy of the bar is given by the functional $\alpha \mapsto \tilde{\mathcal{E}}_t(\alpha)$ defined on $\mathcal{D}$ by:

$$\tilde{\mathcal{E}}_t(\alpha) = \frac{E_0\ell^2}{2}\int_0^L \alpha'(x)^2 dx + \int_0^L w(\alpha(x))\, dx + \frac{t^2\varepsilon_1^2 L^2}{2\int_0^L \frac{dx}{E(\alpha(x))}}. \tag{53}$$

Its first directional derivative is the linear form defined on $H^1(0, L)$ by

$$\tilde{\mathcal{E}}_t'(\alpha)(\beta) = \int_0^L \left(E_0\ell^2\alpha'\beta' + w'(\alpha)\beta + \frac{t^2\varepsilon_1^2 L^2}{2\left(\int_0^L \frac{dx}{E(\alpha)}\right)^2}\frac{E'(\alpha)}{E(\alpha)^2}\beta \right) dx. \tag{54}$$

Thus, the variational inequation (48) is equivalent to $\tilde{\mathcal{E}}_t'(\alpha_t)(\beta) \geq 0$ for all $\beta \in \mathcal{D}$. Let α_t be a solution at time t. Differentiating once more, we obtain the damage rate problem

$$\text{Find } \dot{\alpha}_t \in \mathcal{D} \text{ such that}$$
$$\tilde{\mathcal{E}}_t''(\alpha_t)(\dot{\alpha}_t, \beta - \dot{\alpha}_t) + \dot{\tilde{\mathcal{E}}}_t'(\alpha_t)(\beta - \dot{\alpha}_t) \geq 0, \quad \forall \beta \in \mathcal{D} \tag{55}$$

where the second directional derivative $\tilde{\mathcal{E}}_t''(\alpha_t)$ is a bilinear symmetric form defined on $H^1(0, L)^2$ and $\dot{\tilde{\mathcal{E}}}_t'(\alpha_t)$ denotes the partial derivative of $\tilde{\mathcal{E}}_t(\alpha)$ with respect to t at $\alpha = \alpha_t$ (it is a linear form defined on $H^1(0,L)$). The explicit expression of $\tilde{\mathcal{E}}_t''(\alpha_t)$ and $\dot{\tilde{\mathcal{E}}}_t'(\alpha_t)$ will be given in the next section for a particular model.

6. Bifurcation phenomena with gradient damage model

To simplify the presentation and in order to obtain closed form solutions, we will only consider in the sequel the following particular damage model:

$$E(\alpha) = \frac{E_0}{(1+\alpha)^2}, \qquad w(\alpha) = \frac{\sigma_1^2}{E_0}\,\alpha, \qquad \alpha \geq 0, \tag{56}$$

E_0 denoting the initial Young modulus, σ_1 the elastic yield stress and ℓ the internal length of the material.

In this case, the damage yield criterion (50) and the relation (47) become

$$E_0^2\ell^2\alpha_t''(x) + \sigma_t^2(1+\alpha_t(x)) \leq \sigma_1^2, \quad \forall x \in (0,L), \tag{57}$$

$$\sigma_t = \frac{t\sigma_1 L}{\int_0^L (1+\alpha_t(x))^2\,dx}. \tag{58}$$

Moreover the second derivative reads as

$$\tilde{\mathcal{E}}_t''(\alpha_t)(\beta) = E_0\ell^2 \int_0^L \beta'^2\,dx + \frac{4\sigma_t^3}{tE_0\sigma_1 L}\left(\int_0^L (1+\alpha_t)\beta\,dx\right)^2 - \frac{\sigma_t^2}{E_0}\int_0^L \beta^2\,dx, \tag{59}$$

where σ_t denotes the equilibrium stress associated to the homogeneous damage α_t, see (58), and

$$\dot{\tilde{\mathcal{E}}}_t'(\alpha_t)(\beta) = -\frac{2\sigma_t^2}{tE_0}\int_0^L (1+\alpha_t)\beta\,dx. \tag{60}$$

For $t \in [0,1)$, the response of the bar is elastic, the damage field remains at its initial value 0, the inequality in (57) is strict. At time $t=1$, the inequality is an equality at every material point and the damage can evolve everywhere. The goal of this section is to prove that we can construct a continuum of solutions for the damage evolution problem for $t>1$, when the length of the bar is sufficiently large. Let us first note that, in any interval where the damage yield criterion (57) is satisfied as an equality at time t (such points are called damaging points), the damage field is given by

$$\alpha_t(x) = \frac{\sigma_1^2}{\sigma_t^2} - 1 + a_t \sin\frac{\sigma x}{E_0\ell} + b_t \cos\frac{\sigma x}{E_0\ell} \tag{61}$$

where a_t and b_t are two time dependent scalars.

6.1. The homogeneous solution.

The homogeneous solution corresponds to the solution where the damage field is uniform at each time, $\alpha_t(x) = \alpha_t$. This solution always exists and we easily deduce from (57) and (58) that

$$\sigma_t = t^{-1/3}\sigma_1, \qquad \alpha_t = t^{2/3} - 1, \quad \forall t > 1. \tag{62}$$

In a diagram $\sigma - \varepsilon$, σ being the equilibrium stress and ε the overall strain of the bar ($\varepsilon_t = t\varepsilon_1$), the overall response of the bar corresponds to the descending branch in Figure 6.1, showing a *softening* behavior of the material.

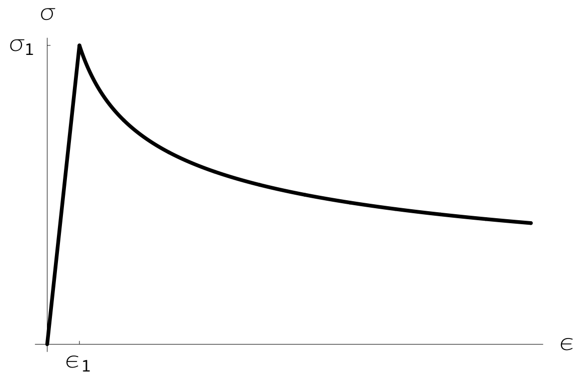

Figure 1. Homogeneous response of the bar: the segment line corresponds to the elastic response ($\alpha_t = 0$, $0 \le t < 1$), the decreasing branch corresponds to a spatially uniform damage growing with time ($\alpha_t(x) = t^{2/3} - 1$).

6.2. Uniqueness criterion

To see whether the solution can be unique, we can use the damage rate problem (55). Let α_t and σ_t be the homogeneous solution at time t of the evolution problem and given by (62). The rate variational problem admits the solution $\dot{\alpha}_t(x) = \frac{2}{3}t^{-1/3}$ for all $x \in (0, L)$ which corresponds to the rate of the homogeneous solution. This will be the unique solution provided $\mathsf{E}_t''(\alpha_t)$ is a definite positive quadratic form on $H^1(0, L)$. Here, the second derivative reads as

$$\tilde{\mathcal{E}}_t''(\alpha_t)(\beta) = E_0\ell^2 \int_0^L \beta'^2\,dx + \frac{4\sigma_t^2}{E_0 L}\left(\int_0^L \beta\,dx\right)^2 - \frac{\sigma_t^2}{E_0}\int_0^L \beta^2\,dx. \tag{63}$$

By introducing the Rayleigh ratio $\mathcal{R}_t$ defined on $H^1(0, L) \setminus \{0\}$ by

$$\mathcal{R}_t(\beta) = \frac{E_0\ell^2 \displaystyle\int_0^L \beta'^2\,dx + \frac{4\sigma_t^2}{E_0 L}\left(\int_0^L \beta\,dx\right)^2}{\displaystyle\frac{\sigma_t^2}{E_0}\int_0^L \beta^2\,dx}, \tag{64}$$

it immediately appears that

> The rate damage problem admits a unique solution if $\min_{H^1(0,L)\setminus\{0\}} \mathcal{R}_t > 1$.

After some calculations, too long to be reported here (see Pham et al. (2011b)[Appendix]), we obtain

$$\min_{H^1(0,L)\setminus\{0\}} \mathcal{R}_t = \min\left\{4, \left(\frac{\sigma_c}{\sigma_t}\right)^2\right\} \tag{65}$$

where

$$\sigma_c = \pi E_0 \frac{\ell}{L}. \tag{66}$$

Thus, we can conclude that

1. *If* $\sigma_1 \leq \sigma_c$, *i.e.* if $L \leq \pi\ell/\varepsilon_1$, then the homogeneous solution is the unique solution of the damage evolution problem. This feature can be used for instance in an experimental setting to identify properly the homogeneous behaviour of the material.
2. *If* $\sigma_1 > \sigma_c$, *i.e.* if $L > \pi\ell/\varepsilon_1$, then the homogeneous damage rate $\dot{\alpha}_t = \frac{2}{3}t^{-1/3}$ is the unique solution of the rate damage problem provided that $\sigma_t < \sigma_c$, *i.e.* when $t > \frac{\sigma_1^3}{\sigma_c^3}$. Bifurcations are *eventually* possible from the homogeneous solution at any time in the interval $[1, \frac{\sigma_1^3}{\sigma_c^3}]$. When available, first bifurcation will occur at $t = 1$.

6.3. Examples of bifurcated branches at t=1

To construct non homogeneous solutions, we can investigate solutions where the equality in (57) holds only in a time-dependent part of the bar, i.e the bar elastically unloads with the initial stiffness E_0 in the rest part. Different scenarii may exist depending on the length of the bar.

For instance, one can assume that the equality holds in the interval $(0, D_t)$ and in this case we obtain the following solution

$$\alpha_t(x) = \begin{cases} 2\Big(\frac{\sigma_1^2}{\sigma_t^2} - 1\Big)\cos^2\frac{\pi x}{2D_t} & , \text{ if } 0 \leq x \leq D_t \\ 0 & , \text{ otherwise} \end{cases}, \tag{67}$$

where the width D_t is related to the stress equilibrium σ_t by

$$D_t = \pi\frac{E_0}{\sigma_t}\ell \tag{68}$$

and the overall response reads as

$$t = \frac{\sigma_t}{\sigma_1} + \frac{\sigma_c}{2\sigma_1}\left(3\frac{\sigma_1^4}{\sigma_t^4} - 2\frac{\sigma_1^2}{\sigma_t^2} - 1\right). \tag{69}$$

This half-sinusoidal damage field can appear provided that the bar is long enough so that $D_1 \leq L$, *i.e.* provided that $\sigma_c < \sigma_1$. The length D_1 is the size of the damaging zone just after $t = 1$. For $t > 1$, the irreversibility condition (49) is satisfied and the damage grows provided that $4\sigma_c > \sigma_1$. So, if

$$\frac{\varepsilon_1}{4} \leq \pi \frac{\ell}{L} \leq \varepsilon_1, \tag{70}$$

the solution is valid as long as $D_t \leq L$, *i.e.* for $t \in [1, t_c]$ with

$$t_c = \frac{\sigma_1^2}{\sigma_c^2} \left(3\frac{\sigma_1^2}{\sigma_c^2} - 2 \right). \tag{71}$$

During this time interval, the damaging zone extends gradually to all the bar, cf. Figure 6.3.

Remark 4. 1. *One can construct symmetrically one half-sinusoidal damage field in the interval* $(L - D_t, L)$. *The global response is the same.*

2. *At* $t = t_c$ *the tip of the damaged zone reaches the end* $x = L$ *and can no more propagate. To continue this branch, we must consider solutions in which a part of the bar is in an unloading phase (the inequality is then strict in* (57)*). The details are not given here.*

3. *If the bar is too small, i.e. if* $L < \pi\ell/\varepsilon_1$, *then the half-sinusoidal damage field cannot appear for lack of place. We recover the uniqueness property that we have obtained from the rate damage problem.*

4. *If the bar is too long, i.e. if* $L > 4\pi\ell/\varepsilon_1$, *then the global response* $\sigma - \varepsilon$ *has a* snap-back *near* $(\varepsilon_1, \sigma_1)$, *i.e.* $\frac{d\sigma}{d\varepsilon}(\varepsilon_1 -) > 0$, *cf. Figure 6.3. So, since the overall strain* $\varepsilon_t = t\varepsilon_1$ *must increase, the stress must brutally decrease and the damage field must brutally increase. The response is no more continuous in time.*

Other similar but qualitatively different solution is when the equality holds in the interval $(x_0 - D_t, x_0 + D_t)$ where D_t is a variable length and x_0 is a given point. We obtain the following solution

$$\alpha_t(x) = \begin{cases} 2\Big(\frac{\sigma_1^2}{\sigma_t^2} - 1\Big) \cos^2 \dfrac{\pi(x - x_0)}{2D_t} & , \text{ if } |x - x_0| \leq D_t \\ 0 & , \text{ otherwise} \end{cases}, \tag{72}$$

with D_t still given by (68), but the overall response reads now as

$$t = \frac{\sigma_t}{\sigma_1} + \frac{\sigma_c}{\sigma_1} \left(3\frac{\sigma_1^4}{\sigma_t^4} - 2\frac{\sigma_1^2}{\sigma_t^2} - 1 \right). \tag{73}$$

This sinusoidal damage field can appear, centered at a given point x_0, provided that the bar is long enough. Let us consider the case where $x_0 = L/2$. The sinusoidal damage field can appear, centered at the middle of the bar, if $2D_1 \leq L$, *i.e.* provided that $2\sigma_c \leq \sigma_1$. The length $2D_1$ is the size of the damaging zone just after $t = 1$. For $t > 1$, the irreversibility condition (49) is satisfied and the damage grows provided that $8\sigma_c \geq \sigma_1$. So, if

$$\frac{\varepsilon_1}{4} \leq 2\pi \frac{\ell}{L} \leq \varepsilon_1, \tag{74}$$

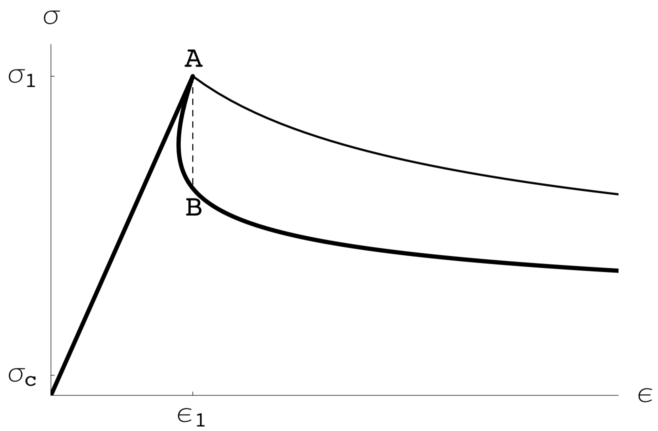

Figure 2. Global response of the bar corresponding to the growing of one half-sinusoidal damage field (thick line) compared to the homogeneous response (thin line). The length is sufficiently large ($L > 4\pi\ell/\varepsilon_1$) and a snap-back is present at the bifurcation point A. Consequently, the initiation of damage will be discontinuous, the global response must jump from A to B or to a point below B at time $t = 1$.

the solution is valid as long as $2D_t \leq L$, *i.e.* for $\sigma_t \in [2\sigma_c, \sigma_1]$. During this time interval, the damaging zone extends gradually to all the bar, cf. Figure 6.3.

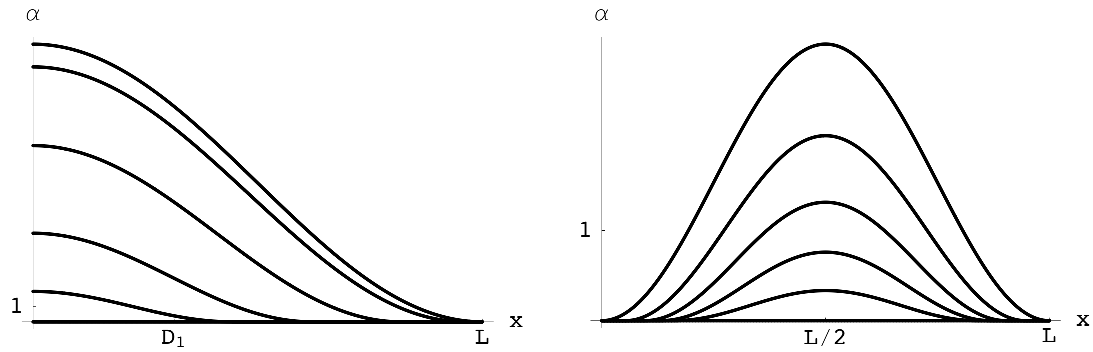

Figure 3. On the left: *Growing of one half-sinusoidal damage field starting at the end $x = 0$. The size of the damaged zone is equal to D_1 at $t = 1+$, then increases progressively with t and all the bar is damaged at $t = t_c$.* On the right: *Growing of one sinusoidal damage field centered at the middle of the bar. The size of the damaging zone is equal to $2D_1$ at $t = 1+$, then increases progressively with t untill all the bar is damaging.*

If the bar is longer, one can construct solutions with n sinusoidal waves and in this case (69) is simply replaced by

$$t = \frac{\sigma_t}{\sigma_1} + n\frac{\sigma_c}{\sigma_1}\left(3\frac{\sigma_1^4}{\sigma_t^4} - 2\frac{\sigma_1^2}{\sigma_t^2} - 1\right). \tag{75}$$

The global responses corresponding to these non homogeneous solutions are plotted in Figure 6.3.

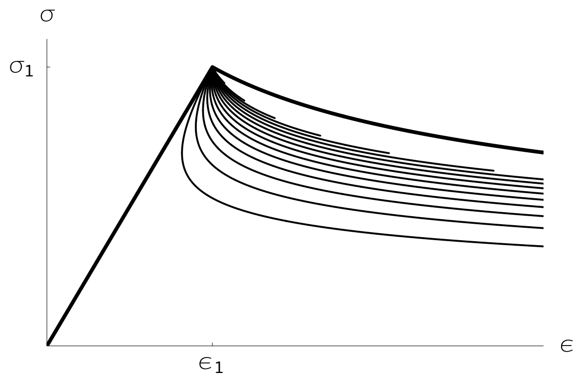

Figure 4. In thin lines, the global response due to the growing of n-sinusoidal damage fields. The lowest curve corresponds to $n = 0.5$, the next to $n = 1$, ... The number of curves depends on the length of the bar: the longer is the bar, the larger is the number of curves. The curve n stops when the n-sinusoidal damage field covers all the bar.

6.4. Bifurcation from the homogeneous branch.

We propose to construct in this subsection a continuum of solutions for the damage evolution problem by considering bifurcation branches from any point of the homogeneous one when the length of the bar is sufficiently large. We assume that

$$L > \pi\ell/\varepsilon_1 \tag{76}$$

and we proceed as follows

1. Let $\alpha_b > 0$ be a given value of the damage variable. In the case of the homogeneous response, the corresponding time at which this damage state is reached, the corresponding overall strain and the corresponding equilibrium stress are given by

$$t_b = (1+\alpha_b)^{3/2}, \quad \varepsilon_b = (1+\alpha_b)^{3/2}\varepsilon_1, \quad \sigma_b = \frac{\sigma_1}{\sqrt{1+\alpha_b}}. \tag{77}$$

2. For $t > t_b$, we seek for a non homogeneous solution, the damage growing in the part $(0, D_t)$ of the bar while the damage remains at the value α_b in the remainder part (D_t, L) of the bar. By using (57)–(61) we find

$$\alpha_t(x) = \begin{cases} \alpha_b + 2\Big(\frac{\sigma_1^2}{\sigma_t^2} - 1 - \alpha_b\Big)\cos^2\frac{\pi x}{2D_t} & , \text{ if } 0 \le x \le D_t \\ \alpha_b & , \text{ otherwise} \end{cases}, \tag{78}$$

with

$$D_t = \pi\frac{E_0}{\sigma_t}\ell \tag{79}$$

and

$$t = (1+\alpha_b)^2 \frac{\sigma_t}{\sigma_1} + \frac{\sigma_c}{2\sigma_1}\left(3\frac{\sigma_1^4}{\sigma_t^4} - 2(1+\alpha_b)\frac{\sigma_1^2}{\sigma_t^2} - (1+\alpha_b)^2\right). \tag{80}$$

The damage field corresponds to the growing of one half-sinusoid from the initial value α_b, cf. Figure 6.4. This solution is valid as long as $D_t \leq L$, *i.e.* as long as $\sigma_t \geq \sigma_c$. So the value of the initial damage α_b must be chosen arbitrarily provided that $\sigma_b > \sigma_c$. Accordingly, by taking $\alpha_b \in [0, \frac{\sigma_1^2}{\sigma_c^2} - 1)$, we have obtained an infinite family of solutions indexed by α_b, cf. Figure 6.4.

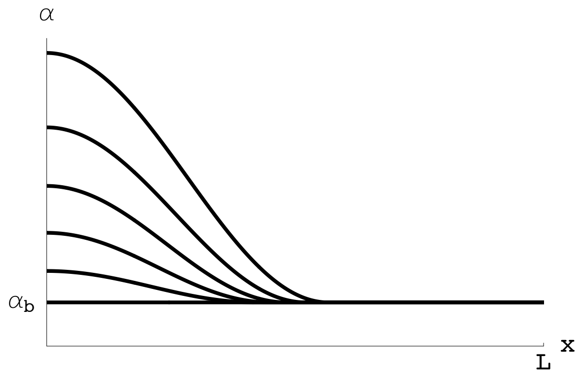

Figure 5. Growing of one half-sinusoidal damage field from an initially homogeneous damage state α_b.

Remark 5. *These results reinforce those we have obtained from the rate damage problem, since they prove that a bifurcation from the homogeneous branch is* really *possible at any time in the interval* $[1, \frac{\sigma_1^3}{\sigma_c^3}]$.

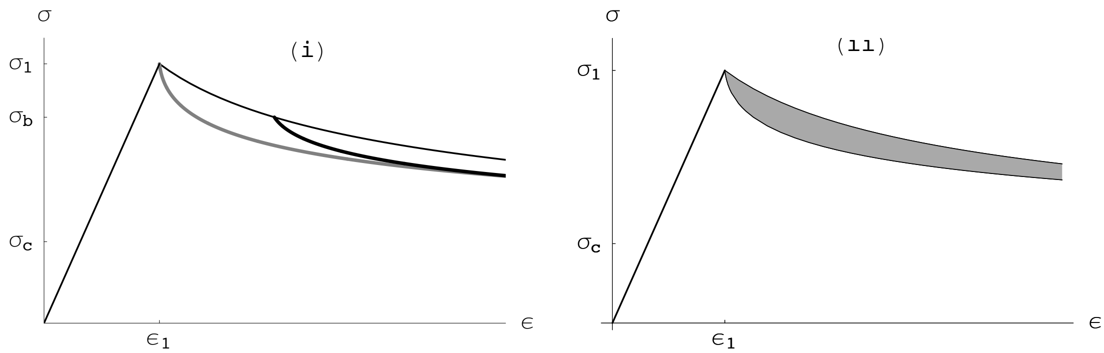

Figure 6. (i) *Thick black line:* the bifurcated branch; *Thin line:* the homogeneous branch; *Thick gray line:* the one half-sinusoidal damage branch. (ii) *Gray area:* the continuum of possible global responses due to the possibility of a bifurcation at any point of the homogeneous branch.

7. Stability and selection of solutions

7.1. Stability criterion

The previous analysis shows that the damage evolution problem is ill-posed in the sense that it admits a continuum family of solutions . Therefore, the question of selection of the solutions raises in order to choose those solutions that can be really observed. We suggest here to answer this question by stability considerations and postulate that the solutions that are potentially feasible are the stable ones. And more precisely, those solutions that minimize the total energy. Following the idea presented in part I, we can use the stability condition (ST) that we briefly recall below[2]. Let α_t be an admissible damage field and let u_t be the associated displacement field giving the equilibrium of the bar at time t, *i.e.*

$$E(\alpha_t(x))u_t'(x) = \sigma_t, \quad \forall x \in (0, L), \quad \text{with} \quad \sigma_t = \frac{t\varepsilon_1 L}{\int_0^L \frac{dx}{E(\alpha_t(x))}}. \tag{81}$$

The total energy of the bar is given by the functional $\alpha \mapsto \tilde{\mathcal{E}}_t(\alpha)$ defined on $\mathcal{D}$ by (53). We say that the bar is *stable* at time t in its damaged state α_t if and only if α_t is a unilateral local minimum of $\tilde{\mathcal{E}}_t$ on $\mathcal{D}$, *i.e.*

$$(\mathsf{ST}) \qquad \forall \beta \in \mathcal{D}, \quad \exists \bar{h} > 0 \text{ such that } \forall h \in [0, \bar{h}], \quad \tilde{\mathcal{E}}_t(\alpha_t) \le \tilde{\mathcal{E}}_t(\alpha_t + h\beta).$$

Let us note that we have only to compare the energy of α_t with the energy of the damage states which are accessible from α_t. This unilateral restriction is due to the irreversibility condition. This condition means that, if we can find in a neighborhood of α_t an accessible damage state with a smaller energy, then the state α_t is unstable and the bar will evolve spontaneously to some state with a smaller energy.

7.2. Stability of the homogeneous states

For illustration, we will only study the stability of the homogeneous states of the bar, the analysis can be extended to the other solutions but is too long to be reported here. Let $\alpha_t = t^{2/3} - 1$ be the homogeneous damage state of the bar at the time $t > 1$, $h > 0$ and $\beta \in \mathcal{D}$. By developing $\tilde{\mathcal{E}}_t(\alpha_t + h\beta)$ with respect to h, we get

$$\tilde{\mathcal{E}}_t(\alpha_t + h\beta) = \tilde{\mathcal{E}}_t(\alpha_t) + h\tilde{\mathcal{E}}_t'(\alpha_t)(\beta) + \tfrac{1}{2}h^2\tilde{\mathcal{E}}_t''(\alpha_t)(\beta) + o(h^2), \tag{82}$$

where the primes denote directional derivatives. Since $\tilde{\mathcal{E}}_t'(\alpha_t)(\beta) = 0$, the stability condition consists in finding the sign of the second derivative in any positive direction β. In the particular

[2] Here the stability condition is formulated in terms of the damage field alone because the displacement field has been eliminated by using the equilibrium equation.

Table I. Stability of an homogeneous state and possibility of bifurcation from this state following the value of the equilibrium stress.

Case	Stability	Bifurcation
$\sigma_t > 4\sigma_c$	No	Yes
$\sigma_c < \sigma_t < 4\sigma_c$	Yes	Yes
$\sigma_t < \sigma_c$	Yes	No

case of the model (56), the second derivative is given by (59). By considering the Rayleigh ratio (64), it immediately appears that for the the homogeneous damage state α_t to be stable, it is necessary that $\min_{\mathcal{D}\setminus\{0\}} \mathcal{R}_t \geq 1$. It is sufficient however for this state to be stable that $\min_{\mathcal{D}\setminus\{0\}} \mathcal{R}_t > 1$. After some calculations which are not reproduced here (see Pham et al. (2011b)[Appendix]), we obtain

$$\min_{\mathcal{D}\setminus\{0\}} \mathcal{R}_t = \min\left\{4, \left(\frac{4\sigma_c}{\sigma_t}\right)^{2/3}\right\} \tag{83}$$

and we can conclude that

1. *If* $\sigma_1 \leq 4\sigma_c$, *i.e.* if $L \leq 4\pi\ell/\varepsilon_1$, then all the homogeneous damage states are stable.

2. *If* $\sigma_1 > 4\sigma_c$, *i.e.* if $L > 4\pi\ell/\varepsilon_1$, then only the homogeneous damage states at $t \geq \left(\frac{\sigma_1}{4\sigma_c}\right)^3$ are stable. Consequently, since the beginning of the homogeneous branch is not stable, the bar cannot be deformed uniformly and a non homogeneous solution will appear at $t = 1$.

Let us now compare the properties of stability and of bifurcation that we have obtained for the homogeneous solution. Since $\mathcal{D}$ is included in $H^1(0, L)$, we always have

$$4 \geq \min_{\mathcal{D}\setminus\{0\}} \mathcal{R}_t \geq \min_{H^1(0,L)\setminus\{0\}} \mathcal{R}_t$$

and uniqueness of the rate solution implies stability of the state. But the converse is not always true, a homogeneous state could be stable even if a bifurcation is possible at this point. Let us examine the different cases, α_t and σ_t are still given by (62). Now, if we consider bars with different length, we obtain the following scenarii:

1. *Small bars:* $L < \pi\ell/\varepsilon_1$. All the homogenous sates are stable and no bifurcation is possible. The homogeneous response is the unique solution of the damage evolution problem.

2. *Intermediate bars:* $\pi\ell/\varepsilon_1 < L < 4\pi\ell/\varepsilon_1$. All the homogenous sates are stable, but bifurcations are possible in the time interval $[1, \frac{\sigma_1^3}{\sigma_c^3}]$.

3. *Long bars:* $L > 4\pi\ell/\varepsilon_1$. Since the homogeneous states are unstable for $t \in [1, \frac{\sigma_1^3}{64\sigma_c^3})$, a non homogeneous solution appears at $t = 1$.

8. Conclusions

A bifurcation and stability analysis was undertaken here for a simple gradient damage model in one-dimensional situation. The full nonlinear initial value problem was solved in closed form for a bar with a finite length. A uniqueness criterion was obtained as well as conditions for bifurcation. These are mainly dependent on the ratio L/ℓ between the length of the bar to the internal lengthscale involved in the model. The longer the bar (or the smaller the lengthscale), the more solutions are obtained. The localization zone, represented here by the damaged zone, has always a finite thickness. However, in contrast to the underlying local model where all the damaged states are shown to be unstable, we have shown the existence of stable states and paths for the gradient model and we suggested that these stable paths can be selected (among all the solutions) as potential responses of the bar.

How these stability and uniqueness properties can be used to identify the state functions $\alpha \mapsto E(\alpha)$, $\alpha \mapsto w(\alpha)$ and the internal length ℓ characterizing the material behavior is explained in Pham et al. (2011b). Extension of the results to three-dimensional situations and their links with the Hill's general theory of uniqueness and stability (Hill, 1958) can be found in Pham and Marigo (2013b).

Part III

Damage localization and crack nucleation

Abstract

In this part we construct solutions with damage localization until rupture for the traction problem of a bar made of a strongly brittle material. We show that damage localization necessarily develops on parts of the bar whose length is proportional to the material internal length and with a profile which is also a material characteristic. From its onset until the rupture, the damage profile is obtained either in a closed form or after a simple numerical integration. Thus, the proposed method provides definitions for the critical stress and fracture energy that can be compared with experimental results. The matter of this part is essentially borrowed from Pham and Marigo (2013a). A similar analysis can also be found in Pham et al. (2011b).

9. Setting of the damage problem

9.1. The gradient damage model

We consider a one-dimensional gradient damage model in which the damage variable α is a real number growing from 0 to 1, where $\alpha = 0$ is the undamaged state and $\alpha = 1$ is the full damaged state. The behavior of the material is characterized by the state function W which gives the energy density at each point x. It depends on the local strain $\boldsymbol{\varepsilon}(x)$ (if u denotes the displacement field, then $\boldsymbol{\varepsilon}(x) = u'(x)$ where the prime stands for the spatial derivative), the local damage value $\alpha(x)$ and the local gradient $\alpha'(x)$ of the damage field at x. Specifically, we assume that W takes the following form

$$\mathsf{W}(\boldsymbol{\varepsilon}, \alpha, \alpha') = \frac{1}{2}E(\alpha)\boldsymbol{\varepsilon}^2 + w(\alpha) + \frac{1}{2}w_1\ell^2\alpha'^2. \tag{84}$$

In (84), $w_1 := w(1)$ represents the energy dissipated in completely damaged volume element and $E(\alpha)$ the Young modulus of the material in the damage state α. The second term $w(\alpha)$ can be interpreted as the density of energy dissipated by the material during a homogeneous damage process (i.e. a process such that $\alpha'(x) = 0$) during which the damage variable grows from 0 to α. The last term in (84) is the "non local" part of the energy which plays, as we will see in the next section, a regularizing role by limiting the possibilities of localization of the damage field. For obvious reasons of physical dimension, this term involves a material characteristic length ℓ which will give the size of the damage localization zone. Denoting by σ the stress, the stress-strain relation reads as

$$\sigma = E(\alpha)\boldsymbol{\varepsilon}. \tag{85}$$

The expression of the energy density (84) is analog to the one proposed by (Comi and Perego, 2001). It implicitly assumes a symmetric behavior in tension and compression. It must be modified to take into account asymmetric behaviors, like in (Comi, 2001) or (Pham and Marigo, 2010a; Pham and Marigo, 2010b). While remaining within the framework of symmetric behavior, a model which can seem more general would consist in replacing the constant $w_1\ell^2$ by a function of α. It can be shown in fact that after an adequate change of variable, the damage parameter α can always be chosen so that the function becomes a constant, see Pham and Marigo (2010b). We thus assume here that the damage parameter has been chosen to this end. Note that another choice was made in Pham et al. (2011b).

The qualitative properties of the (gradient or local) model, in particular its softening or hardening character, strongly depend on some properties of the stiffness function $\alpha \mapsto E(\alpha)$, the dissipation function $\alpha \mapsto w(\alpha)$, the compliance function $\alpha \mapsto S(\alpha) = 1/E(\alpha)$ and their derivatives. From now on we will adopt the following hypothesis:

Hypothesis 1 (Strongly brittle materials)**.** *$\alpha \mapsto E(\alpha)$ and $\alpha \mapsto w(\alpha)$ are non negative and (at least) twice continuously differentiable functions on $[0, 1)$ such that*

$$E(0) = E_0 > 0, \quad E'(\alpha) < 0, \quad E(1) = 0, \tag{86}$$

$$w(0) = 0, \quad w'(\alpha) > 0, \quad w(1) < +\infty, \tag{87}$$

$$\alpha \mapsto -w'(\alpha)/E'(\alpha) \text{ is non decreasing,} \tag{88}$$

$$\alpha \mapsto w'(\alpha)/S'(\alpha) \text{ is decreasing to 0.} \tag{89}$$

This corresponds to the family of strongly brittle materials with softening defined in Pham et al. *(*2011b*).*

Let us comment this Hypothesis before giving an example

1. The interval of definition of α can always be taken as $[0, 1]$ after a change of the damage variable;

2. The condition $E' < 0$ denotes the decrease of the material stiffness when the damage grows;

3. The condition $E(1) = 0$ ensures the total loss of stiffness when $\alpha = 1$;

4. The positivity and the monotonicity of w is natural since $w(\alpha)$ represents the energy dissipated during a damage process where the damage grows homogeneously in space from 0 to α;

5. The boundedness of w is characteristic of strongly brittle materials with softening; this condition disappears in the case of weakly brittle materials with softening or in the case of brittle materials with hardening;

   Remark 6. *This condition plays an essential role in order to construct damage localization up to the rupture of the bar. It was not satisfied in the model used in part II which belongs to the family of weakly brittle damage models in the sense of* Pham et al. *(*2011b*).*

6. The condition of monotonicity of w'/E' is introduced for the sake of simplicity and is unessential. In a *homogeneous* strain and damage response, it denotes that the strain does not decrease when the damage grows, see Section 2.3. This refers to the *strain-hardening* property;

7. The condition of monotonicity of w'/S' is essential; it denotes the *softening* property. In a *homogeneous* strain and damage response, this property leads to the decreasing of the corresponding stress when the damage grows, see Section 2.3;

8. The condition $\lim_{\alpha\to 1} w'(\alpha)/S'(\alpha) = 0$ ensures that the material cannot sustain any stress when its damage state is 1.

Example 5. *A family of models which satisfy the assumptions above is the following one, when $q > p > 0$:*

$$E(\alpha) = E_0(1-\alpha)^q, \quad w(\alpha) = \frac{q\sigma_c^2}{2pE_0}\big(1-(1-\alpha)^p\big). \tag{90}$$

It contains five material parameters: the sound Young modulus $E_0 > 0$, the dimensionless parameters p and q, the critical stress $\sigma_c > 0$ and the internal length $\ell > 0$ whose physical interpretation will be given in Section 10.2.

The condition $q > 0$ is necessary and sufficient in order that $\alpha \mapsto E(\alpha)$ be decreasing from E_0 to 0 while the condition $p > 0$ is necessary and sufficient in order that $\alpha \mapsto w(\alpha)$ be increasing from 0 to a finite value. If $p > 0$ and $q > 0$, then the condition $q > p$ is necessary and sufficient in order that $\alpha \mapsto -w'(\alpha)/E'(\alpha)$ be increasing to ∞ while $\alpha \mapsto w'(\alpha)/S'(\alpha)$ is automatically decreasing to 0.

A particularly interesting class of materials which satisfy the conditions of Hypothesis 1 is that of *perfectly brittle materials* which corresponds to the models in which $p = q > 0$ in the previous example. By definition, these materials are such that w'/E' is constant. This leads to the following definition:

Hypothesis 2 (Perfectly brittle materials)**.** *They are strongly brittle materials such that w'/E' is constant. Accordingly, these materials are characterized by the unique state function $\alpha \mapsto E(\alpha)$ which is twice continuously differentiable and must satisfy*

$$E(0) = E_0 > 0, \qquad E'(\alpha) < 0 \quad \forall\alpha \in [0,1), \qquad E(1) = 0, \tag{91}$$

while $w(\alpha)$ is given by

$$w(\alpha) = (E_0 - E(\alpha))\frac{\varepsilon_c^2}{2}, \tag{92}$$

where ε_c is a given positive constant.

9.2. The damage problem of a bar under traction

Let us consider a homogeneous bar whose natural reference configuration is the interval $(0, L)$ and whose cross-sectional area is 1. The bar is made of the nonlocal damaging material characterized by the state function W given by (84). The end $x = 0$ of the bar is fixed, while the displacement of the end $x = L$ is prescribed to a value U_t

$$u_t(0) = 0, \qquad u_t(L) = U_t \geq 0, \qquad t \geq 0 \tag{93}$$

where, in this quasi-static setting, t denotes the loading parameter or shortly the "time", and u_t is the displacement field of the bar at time t. The dependence of U_t on t is assumed to be smooth, at least continuous and piecewise continuously differentiable. The evolution of displacement and of damage in the bar is obtained using a variational formulation, the main ingredients of which are recalled hereafter, see Benallal and Marigo (2007) for details and Pham and Marigo, Pham and Marigo (2010a, 2010b), Pham et al. (2011a) and Pham et al. (2011b) for a general discussion on the variational formulation of damage evolution problems.

For a given $U \in \mathbb{R}$, we denote by $\mathcal{C}_U$ the set of "smooth" fields v defined on $[0, L]$ and such that $v(0) = 0$, $v(L) = U$, *i.e.*

$$\mathcal{C}_U = \left\{ v \in H^1(0, L) : v(0) = 0,\ v(L) = U \right\} \tag{94}$$

where $H^1(0, L)$ denotes the usual Sobolev space of functions which belong to $L^2(0, L)$ and whose distributional first derivative also belongs to $L^2(0, L)$. Accordingly, $\mathcal{C}_{U_t}$ and $\mathcal{C}_{\dot{U}_t}$ denote the sets of kinematically admissible displacement fields and kinematically admissible displacement rate fields, while $\mathcal{C}_0$ is their associated linear space. The set of admissible damage fields is the convex set $\mathcal{D}$ defined by

$$\mathcal{D} = \left\{ \beta \in H^1(0, L) : 0 \leq \beta(x) < 1,\ \forall x \in [0, L] \right\}. \tag{95}$$

Let us note that the value 1 for the damage is excluded because some quantities like the compliance S and its derivatives are no more defined when $\alpha = 1$. It turns out also that the real displacement field is no more regular but is discontinuous at points x where $\alpha(x) = 1$. Since this situation corresponds to the rupture of the bar, we will merely determine at which time t_r that happens and the analysis will stop at this moment.

By virtue of the irreversibility condition, damage can only grow and accordingly the convex cone $\dot{\mathcal{D}}$ of admissible damage rate is given by

$$\dot{\mathcal{D}} = \left\{ \beta \in H^1(0, L) : \beta(x) \geq 0,\ \forall x \in [0, L] \right\}. \tag{96}$$

With any admissible pair (u, α), we associate the total energy of the bar

$$\begin{aligned} \mathcal{E}(u, \alpha) &:= \int_0^L \mathsf{W}(u'(x), \alpha(x), \alpha'(x))\, dx \\ &= \int_0^L \left(\frac{1}{2} E(\alpha(x)) u'(x)^2 + w(\alpha(x)) + \frac{1}{2} w_1 \ell^2 \alpha'(x)^2 \right) dx \end{aligned} \tag{97}$$

We are in a position to set the evolution problem. Specifically, for a given initial damage field α^0, the damage evolution problem reads as (see part II, Proposition 5.1):

PB 5 (Variational damage evolution problem)**.** *Find $t \mapsto (u_t, \alpha_t)$ absolutely continuous and such that*

1. *For all $t \geq 0$,* $\quad (u_t, \alpha_t) \in \mathcal{C}_{U_t} \times \mathcal{D}$,
2. *For almost all $t > 0$,* $\quad (\dot{u}_t, \dot{\alpha}_t) \in \mathcal{C}_{\dot{U}_t} \times \dot{\mathcal{D}}$,
3. *For almost all $t > 0$ and for all $(v, \beta) \in \mathcal{C}_{\dot{U}_t} \times \dot{\mathcal{D}}$,* $\quad \mathcal{E}'(u_t, \alpha_t)(v - \dot{u}_t, \beta - \dot{\alpha}_t) \geq 0$,

with the initial condition $\alpha_0(x) = \alpha^0(x)$.

In the third item, $\mathcal{E}'(u, \alpha)(v, \beta)$ denotes the derivative of $\mathcal{E}$ at (u, α) in the direction (v, β) and is given by

$$\mathcal{E}'(u, \alpha)(v, \beta) = \int_0^L \left(E(\alpha) u' v' + \left(\frac{1}{2} E'(\alpha) u'^2 + w'(\alpha) \right) \beta + w_1 \ell^2 \alpha' \beta' \right) dx$$

Remark 7. *Note that the second item contains the irreversibility condition $\dot{\alpha}_t \geq 0$ and that our formulation makes sense only if the evolution is sufficiently smooth in time. Therefore, we only consider evolutions such that the displacement field and the damage field are absolutely continuous functions of time. To enlarge the search to evolutions which are discontinuous in time, which is often necessary as we will see in the last section, one has to reformulate the evolution problem and replace the third item by a condition which remains meaningful for discontinuous evolutions. This is the essence of the variational formulation proposed in the first part of these Lecture Notes where the third item of* **PB 5** *is replaced by a stability condition and an energy balance.*

10. Preliminary properties of the traction problem

10.1. The local conditions

Choosing $\beta = \dot{\alpha}_t$ and $v = \dot{u}_t + v_0$ with $v_0 \in \mathcal{C}_0$ and inserting into the third item of **PB 5**, we obtain the variational formulation of the equilibrium of the bar,

$$\int_0^L E(\alpha_t(x)) u_t'(x) v_0'(x)\, dx = 0, \qquad \forall v_0 \in \mathcal{C}_0 \tag{98}$$

From (98), we deduce that the stress is constant all along the bar and hence is only a function of time

$$\sigma_t = E(\alpha_t(x)) u_t'(x), \quad \forall x \in (0, L) \tag{99}$$

Dividing (99) by $E(\alpha_t(x))$, integrating over $(0, L)$ and using boundary conditions (93), we find

$$\sigma_t \int_0^L S(\alpha_t(x))dx = U_t, \tag{100}$$

which gives the overall force-displacement response of the bar once the damage field is known.

To obtain the damage problem which governs the evolution of the damage field in the bar, one inserts (98)–(100) into the third item of **PB 5**. This leads to the variational inequality governing the evolution of the damage

$$\int_0^L \Big(2w'(\alpha_t) - \sigma_t^2 S'(\alpha_t)\Big)(\beta - \dot\alpha_t)\, dx + \int_0^L 2w_1\ell^2\alpha_t'(\beta' - \dot\alpha_t')dx \geq 0 \tag{101}$$

where the inequality must hold for all $\beta \in \dot{\mathcal{D}}$ and almost all $t \geq 0$. Integrating by parts the second integral in (101) leads to a new form of the variational inequality:

$$\int_0^L \Big(2w'(\alpha_t) - \sigma_t^2 S'(\alpha_t) - 2w_1\ell^2\alpha_t''\Big)(\beta - \dot\alpha_t)\, dx + 2w_1\ell^2\Big(\alpha_t'(L)\big(\beta(L) - \dot\alpha_t(L)\big) - \alpha_t'(0)\big(\beta(0) - \dot\alpha_t(0)\big)\Big) \geq 0. \tag{102}$$

Setting first $\beta = 0$ and then $\beta = 2\dot\alpha_t$ in (102), we obtain the equality:

$$\int_0^L \Big(2w'(\alpha_t) - \sigma_t^2 S'(\alpha_t) - 2w_1\ell^2\alpha_t''\Big)\dot\alpha_t\, dx + 2w_1\ell^2\Big(\alpha_t'(L)\dot\alpha_t(L) - \alpha_t'(0)\dot\alpha_t(0)\Big) = 0. \tag{103}$$

Inserting this equality into (102) leads to the following inequality:

$$\int_0^L \Big(2w'(\alpha_t) - \sigma_t^2 S'(\alpha_t) - 2w_1\ell^2\alpha_t''\Big)\beta\, dx + 2w_1\ell^2\Big(\alpha_t'(L)\beta(L) - \alpha_t'(0)\beta(0)\Big) \geq 0, \quad \forall \beta \in \dot{\mathcal{D}}. \tag{104}$$

Choosing first $\beta \in \mathcal{C}_0^\infty(0, L) \cap \dot{\mathcal{D}}$, where $\mathcal{C}_0^\infty(0, L)$ denotes the space of indefinitely differentiable functions with compact support in $(0, L)$, the inequality (103) becomes

$$\int_0^L \Big(2w'(\alpha_t) - \sigma_t^2 S'(\alpha_t) - 2w_1\ell^2\alpha_t''\Big)\beta\, dx \geq 0, \qquad \forall \beta \in \mathcal{C}_0^\infty(0, L), \quad \beta \geq 0,$$

from which we deduce by standard arguments that the following inequality must hold almost everywhere in $(0, L)$:

$$2w'(\alpha_t) - \sigma_t^2 S'(\alpha_t) - 2w_1\ell^2\alpha_t'' \geq 0. \tag{105}$$

Choosing now $\beta(x) = (1 - x/h)^+$ in (104) with $0 < h < L$, $a^+ = \max\{0, a\}$ denoting the positive part of a, one gets

$$\int_0^h \Big(2w'(\alpha_t) - \sigma_t^2 S'(\alpha_t) - 2w_1\ell^2\alpha_t''\Big)\left(1 - \frac{x}{h}\right) dx - 2w_1\ell^2\alpha_t'(0) \geq 0.$$

Passing to the limit when h goes to 0, one obtains

$$\alpha_t'(0) \leq 0. \tag{106}$$

In the same way, choosing $\beta(x) = (1 - (L-x)/h)^+$ one gets

$$\alpha_t'(L) \geq 0. \tag{107}$$

Hence, (102) is satisfied only if (105)–(107) are satisfied. Conversely, one immediately sees that if (105)–(107) are satisfied, then (102) is also satisfied. Consequently, (102) and (105)–(107) are equivalent.

Using (105)–(107) and taking into account the irreversibility condition $\dot\alpha_t \geq 0$, (103) gives the following equalities:

$$\Big(2w'(\alpha_t) - \sigma_t^2 S'(\alpha_t) - 2w_1\ell^2\alpha_t''\Big)\dot\alpha_t = 0, \qquad \alpha_t'(0)\dot\alpha_t(0) = 0, \qquad \alpha_t'(L)\dot\alpha_t(L) = 0, \tag{108}$$

where the first one must hold almost everywhere in $(0, L)$. Finally, one has obtained the following fundamental local version of the evolution problem **PB 5**:

Proposition 10.1. *The pair of absolute continuous functions of time* $t \mapsto (u_t, \alpha_t) \in \mathcal{C}_{U_t} \times \mathcal{D}$ *is solution of* **PB 5** *if and only if, for almost all* $t \geq 0$*, the following conditions hold true*

1. **Equilibrium** *:* $u_t(x) = \sigma_t \int_0^x S(\alpha_t(y))dy$ *and* $U_t = \sigma_t \int_0^L S(\alpha_t(y))dy,$
2. **Irreversibility** *:* $\dot\alpha_t \geq 0$ *a.e. in* $(0, L)$,
3. **Damage criterion in the bulk** *:* $2w'(\alpha_t) - \sigma_t^2 S'(\alpha_t) - 2w_1\ell^2\alpha_t'' \geq 0$ *a.e. in* $(0, L)$,
4. **Consistency condition in the bulk***:* $\Big(2w'(\alpha_t) - \sigma_t^2 S'(\alpha_t) - 2w_1\ell^2\alpha_t''\Big)\dot\alpha_t = 0$ *a.e. in* $(0, L)$,,
5. **Damage boundary condition** *:* $\alpha_t'(0) \leq 0$ *and* $\alpha_t'(L) \geq 0$,
6. **Consistency condition at the boundary** *:* $\alpha_t'(0)\dot\alpha_t(0) = 0$ *and* $\alpha_t'(L)\dot\alpha_t(L) = 0$.

Remark 8. *We have implicitly assumed that* $x \mapsto \alpha_t(x)$ *is a sufficiently smooth field so that the integration by parts which leads to* (102) *is licit. The damage criterion in the bulk* (105) *makes sense provided that* α_t *is at least continuously differentiable. Such a regularity result could be obtained after a careful treatment of the variational inequality* (101)*, but it is outside the scope of the present paper and this regularity property will be admitted.*

Remark 9. *From the variational approach, we have deduced boundary conditions for the damage field. These* natural *boundary conditions are due to the fact that no* a priori *restrictions are imposed to the damage at the boundaries. Of course, these boundary conditions disappear if we assume that the end points of the bar cannot be damaged. In such a case, the sets of admissible damage fields and of admissible rate damage fields become*

$$\mathcal{D} = \{\alpha \in H_0^1(0, L) \,:\, 0 \leq \alpha < 1 \text{ in } [0, L]\}, \qquad \dot{\mathcal{D}} = \{\alpha \in H_0^1(0, L) \,:\, \alpha \geq 0 \text{ in } [0, L]\}$$

and the conditions $\alpha_t(0) = \alpha_t(L) = 0$, $\dot\alpha_t(0) = \dot\alpha_t(L) = 0$ *replace the items 5 and 6 in the setting of the evolution problem above. More generally, a large variety of boundary conditions can be* *considered in our variational approach. In any case, by duality, a natural condition is associated with each degree of freedom left by the evolution of damage at the boundary. It is one of the numerous advantages of the variational approach. In the present paper, since no restrictions are imposed to the damage at the boundaries, the homogeneous response is possible whereas the response is necessarily non homogeneous if one constraints the ends to remain undamaged. However, the construction of the localized solution inside the bar does not depend on the damage boundary conditions.*

In terms of energy, we have the following property

Property 10.2 (Balance of energy)**.** *Let us assume that the bar is undamaged and unstretched at time 0, i.e.* $\alpha_0 = 0$ *and* $U_0 = 0$*. By definition, the work done by the external loads up to time* t *is given by*

$$\mathcal{W}_e(t) = \int_0^t \sigma_s \dot{U}_s ds, \tag{109}$$

the total dissipated energy in the bar during the damage process up to time t *is given by*

$$\mathcal{E}_d(t) = \frac{1}{2} w_1 \ell^2 \int_0^L \alpha_t'(x)^2 dx + \int_0^L w(\alpha_t(x)) dx, \tag{110}$$

while the elastic energy which remains stored in the bar at time t *is equal to*

$$\mathcal{E}_e(t) = \frac{\sigma_t^2}{2} \int_0^L S(\alpha_t(x)) dx. \tag{111}$$

By virtue of the conditions of **PB 10.1** *that the fields have to satisfy, the following balance of energy holds true at each time:*

$$\mathcal{W}_e(t) = \mathcal{E}_e(t) + \mathcal{E}_d(t).$$

Proof. By virtue of the equilibrium condition and by definition of the elastic energy, the work done by the external load can read as

$$\begin{aligned} \mathcal{W}_e(t) &= \int_0^t \sigma_s \left(\dot{\sigma}_s \int_0^L S(\alpha_s(x)) dx + \sigma_s \int_0^L S'(\alpha_s(x)) \dot{\alpha}_s(x) dx \right) ds \\ &= \int_0^t \dot{\mathcal{E}}_e(s) ds + \int_0^t \int_0^L \frac{\sigma_s^2}{2} S'(\alpha_s(x)) \dot{\alpha}_s(x) dx\, ds. \end{aligned}$$

Using the initial condition and the consistency condition in the bulk, one gets

$$\begin{aligned} \mathcal{W}_e(t) &= \mathcal{E}_e(t) + \int_0^t \int_0^L w'(\alpha_s) \dot{\alpha}_s \, dx\, ds - w_1 \ell^2 \int_0^t \int_0^L \alpha_s'' \dot{\alpha}_s \, dx\, ds \\ &= \mathcal{E}_e(t) + \int_0^L w(\alpha_t) dx + w_1 \ell^2 \int_0^t \int_0^L \alpha_s' \dot{\alpha}_s' \, dx\, ds - w_1 \ell^2 \int_0^t \Big(\alpha_s'(L) \dot{\alpha}_s(L) - \alpha_s'(0) \dot{\alpha}_s(0) \Big) ds. \end{aligned}$$

Using once more the initial condition and the consistency condition at the boundary, we obtain the desired equality. □

10.2. The homogeneous solution and the issue of uniqueness

If we assume that the bar is undamaged at $t = 0$, *i.e.* if $\alpha^0(x) = 0$ for all x, then it is easy to check that the damage evolution problem admits a solution where α_t depends on t but not on x. This particular solution will be called the *homogeneous solution*. Let us construct it in the case where the prescribed displacement is monotonically increasing, *i.e.* when $U_t = tL$ and under the stronger assumption that $\alpha \mapsto -w'(\alpha)/E'(\alpha)$ is increasing from a positive value to $+\infty$ when α grows from 0 to 1 (instead of being merely non decreasing as it is stated in Hypothesis 1, see comment 6).

Since we assume spatial homogeneity for α_t, we have $u_t(x) = tx$ and it remains to find the two time functions $t \mapsto \alpha_t$ and $t \mapsto \sigma_t$. From (100), we get $\sigma_t = E(\alpha_t)t$. Inserting this relation into (105) and (108) leads to

$$\frac{t^2}{2} \leq -\frac{w'(\alpha_t)}{E'(\alpha_t)}, \qquad \dot{\alpha}_t \left(\frac{t^2}{2} + \frac{w'(\alpha_t)}{E'(\alpha_t)}\right) = 0. \tag{112}$$

Since $\alpha_0 = 0$, the inequality for the damage criterion in (112) is strict at $t = 0$ and hence by continuity during a certain time interval. During this time interval, the bar remains undamaged by virtue of the consistency condition in (112). Hence $\alpha_t = 0$ holds as long as the inequality in (112) remains strict. Therefore α_t remains equal to 0 as long as $t \leq \varepsilon_c$ with

$$\varepsilon_c := \sqrt{\frac{2w'(0)}{-E'(0)}}. \tag{113}$$

This corresponds to the elastic phase. For $t > \varepsilon_c$, since $-w'/E'$ is assumed to be increasing, the first relation of (112) must be an equality. Therefore α_t is given by

$$\alpha_t = \left(-\frac{w'}{E'}\right)^{-1} \left(\frac{t^2}{2}\right) \tag{114}$$

and grows from 0 to 1 when t grows from ε_c to ∞. During this damaging phase, the stress σ_t is given by

$$\sigma_t = \sqrt{\frac{2w'(\alpha_t)}{S'(\alpha_t)}}. \tag{115}$$

Since w'/S' is decreasing to 0 by virtue of Hypothesis 1, σ_t decreases from σ_c to 0 when t grows from ε_c to ∞ and the critical stress σ_c is given by

$$\sigma_\mathsf{c} := \sqrt{\frac{2w'(0)}{S'(0)}} = E_0\varepsilon_c. \tag{116}$$

This last property corresponds to the softening character of the damage model as it was announced in comment 7 after Hypothesis 1. Note that σ_t tends only asymptotically to 0, which means that an infinite displacement is necessary to break the bar in the case of a homogeneous response. The damage rate and the stress rate are discontinuous at $t = \varepsilon_c$. Indeed, just before t_c, one has $\dot{\alpha}_t = 0$ and $\dot{\sigma}_t = E_0$, while, just after, one has $\dot{\alpha}_t > 0$ and $\dot{\sigma}_t < 0$. For further comparison with non-homogeneous solutions, let us calculate the stress rate at the beginning of the damage phase. Differentiating (114) with respect to t gives the damage rate $\dot{\alpha}_t$, while differentiating (115) with respect to t gives the stress rate $\dot{\sigma}_t$ in terms of $\dot{\alpha}_t$. Combining both relations finally gives

$$\lim_{t\downarrow\varepsilon_c} \dot{\sigma}_t = -\frac{E_0}{\dfrac{2S'(0)^2{\sigma_\mathsf{c}}^2 E_0}{S''(0){\sigma_\mathsf{c}}^2 - 2w''(0)} - 1}. \tag{117}$$

In terms of energy, the dissipated energy during the damage process is given by

$$\mathcal{E}_d(t) = w(\alpha_t)L.$$

Hence, it is proportional to the length of the bar. The total energy spent to obtain a fully damaged state is equal to $w(1)L$ and hence is finite by virtue of Hypothesis 1.

The non local term has no influence on the homogeneous solution. The length of the bar does not play a role and the homogeneous response is the same whatever the bar length.

Let us now examine the issue of the uniqueness of the response. In the case of local damage models (which are obtained by taking $\ell = 0$), it is well known that the evolution problem admits an infinite number of solution. Does the gradient term ensure the uniqueness? The answer essentially depends on the ratio L/ℓ of the bar length with the internal length, as it is proved in part II in a particular case and in Pham et al. (2011b) in the general case. Specifically it was shown in Pham et al. (2011b)[Proposition 4.4] that a bifurcation from the homogeneous solution is possible at time $t \geq \varepsilon_c$ if and only if $L \geq \mathsf{D}(t)$ with

$$\mathsf{D}(t) = \pi\ell\sqrt{\frac{2w_1}{\sigma_t^2 S''(\alpha_t) - 2w''(\alpha_t)}}.$$

In particular, a bifurcation can occur at the end of the elastic phase and leads to a non homogeneous damage evolution if $L \geq D_c$ with

$$D_c = \pi\ell\sqrt{\frac{2w_1}{{\sigma_\mathsf{c}}^2 S''(0) - 2w''(0)}}. \tag{118}$$

The main goal of the next section is to construct explicitly such a bifurcated solution from the onset of damage to the break of the bar.

We assume throughout this section that the ratio L/ℓ is sufficiently large in order that the boundary conditions at $x = 0$ and $x = L$ do not perturb the construction of the non homogeneous solution.

11. Construction of non homogeneous solutions

Let us consider a solution of the evolution problem. We deduce from (105) that $0 \leq \sigma_t \leq \sigma_\mathsf{c}$. Indeed, $\sigma_t \geq 0$ by virtue of (93) and (100). Then, integrating (105) over $(0, L)$ and using the boundary conditions (106) and (107), we obtain

$$\sigma_t^2 \int_0^L S'(\alpha_t(x))dx \leq \int_0^L 2w'(\alpha_t(x))dx + 2w_1\ell^2(\alpha_t'(0) - \alpha_t'(L)) \leq \int_0^L 2w'(\alpha_t(x))dx. \quad (119)$$

But, since w'/S' is a decreasing function of α by virtue of Hypothesis 1 and since $\alpha_t \geq 0$, we have

$$2w'(\alpha_t(x)) \leq {\sigma_\mathsf{c}}^2 S'(\alpha_t(x)), \quad \forall x \in (0, L).$$

Integrating over $(0, L)$ and inserting the result into (119) gives $\sigma_t^2 \leq {\sigma_\mathsf{c}}^2$. Therefore σ_c is the maximal stress that the material can sustain in any evolution and not only during a homogeneous damage process.

Let us remark that any solution of the evolution problem contains the same elastic phase, *i.e.* $\alpha_t = 0$ as long as U_t remains smaller than $\varepsilon_c L$. Therefore, damage localizations can appear only when U_t has reached the critical value $\varepsilon_c L$ and hence σ_t has reached the critical value σ_c. This critical time is denoted t_c.

The starting point in the construction of non homogeneous solutions is to seek for solutions for which the equality in (105) holds only in some parts of the bar. For a given $t > t_c$, the damage field will be characterized by the set $\mathcal{S}_t = \bigcup_i \mathcal{S}_t^i$ made of a finite number of intervals $\mathcal{S}_t^i$ where $\alpha_t > 0$. In $[0, L]\backslash\mathcal{S}_t$, the material is supposed to be sound and $\alpha_t = 0$. This part of the bar will be called the *(still) elastic part* of the bar while the interval $\mathcal{S}_t^i$ will be called a *(damage) localization zone* and the damage field inside a *(damage) localization profile*. We must discriminate an *inner* localization zone where $\mathcal{S}_t^i$ is an open interval of the form $(x_i - D_t^i, x_i + D_t^i) \subset (0, L)$ from a *boundary* localization zone where $\mathcal{S}_t^i$ is an interval of the form $[0, D_t^i)$ or $(L - D_t^i, L]$. To simplify the presentation, we first consider the inner localization zones. We will indicate after what is changed in the case of a boundary localization zone. The successive steps of the construction are as follows:

1. For a given $t > t_c$, assuming that σ_t is known, we determine the profile of the damage field in a localization zone;
2. We study the dependence of the damage profile on the stress σ_t;
3. We analyze under which condition the irreversibility condition is satisfied.

11.1. Damage profile in a localization zone

Let σ_t be the stress at time $t > t_c$, supposed to be known. We know that $\sigma_{t_c} = \sigma_\mathsf{c}$ and that σ_t cannot be greater than σ_c. The limiting case $\sigma_t = \sigma_c$ will be treated as a particular case and hence one assumes that $\sigma_t < \sigma_\mathsf{c}$. We will see that $\sigma_t = 0$ when the damage field takes the critical value 1 somewhere in the bar. This limiting case will also be treated as a particular case. Accordingly, by continuity, we first consider the cases where $\sigma_t \in (0, \sigma_\mathsf{c})$.

Throughout the remaining part of this subsection and up to the end of the next one, since t is fixed, we omit the index t in all quantities which are time-dependent. We omit also the index i denoting the size D of the considered localization zone (it will appear that this size is, in fact, the same for all localization zones). Let σ be the stress and $\mathcal{S}_i = (x_i - D, x_i + D)$ be a putative inner localization zone. The damage field α must satisfy

$$\alpha > 0 \qquad \text{and} \qquad -\sigma^2 S'(\alpha) + 2w'(\alpha) - 2w_1\ell^2\alpha'' = 0 \quad \text{in} \quad \mathcal{S}_i. \tag{120}$$

Since we assume by construction that the localization zone is matched to an elastic zone where $\alpha = 0$ and since α and α' must be continuous (see Remark 8), the damage field also has to satisfy the boundary conditions

$$\alpha(x_i \pm D) = \alpha'(x_i \pm D) = 0. \tag{121}$$

Multiplying (120) by α' and integrating with respect to x, we obtain the first integral

$$-\sigma^2 S(\alpha) + 2w(\alpha) - w_1\ell^2\alpha'^2 = C \quad \text{in} \quad \mathcal{S}_i, \tag{122}$$

where C is a constant. Using (121) and Hypothesis 1, we get $C = -S_0\sigma^2$ with $S_0 = 1/E_0$ and (122) can read as

$$\ell^2\alpha'(x)^2 = \mathsf{F}(\sigma, \alpha(x)) \quad \text{in} \quad \mathcal{S}_i. \tag{123}$$

In (123), F denotes the function defined in $[0, \sigma_\mathsf{c}] \times [0, 1)$ by

$$\mathsf{F}(\sigma, \beta) := 2\frac{w(\beta)}{w_1} - \frac{\sigma^2}{w_1}\left(S(\beta) - S_0\right). \tag{124}$$

Since $w_1\dfrac{\partial \mathsf{F}}{\partial \beta}(\sigma, \beta) = 2w'(\beta) - \sigma^2 S'(\beta)$ and since, by virtue of Hypothesis 1, $w'(\beta) > 0$ and $1 - \dfrac{\sigma^2 S'(\beta)}{2w'(\beta)}$ decreases from $1 - \sigma^2/\sigma_\mathsf{c}{}^2 > 0$ to $-\infty$ when β grows from 0 to 1, there exists a unique value of β in $(0, 1)$, say $\alpha^*(\sigma)$, for which $\partial\mathsf{F}/\partial\beta$ vanishes:

$$\alpha^*(\sigma) = \left(\frac{w'}{S'}\right)^{-1}\left(\frac{\sigma^2}{2}\right).$$

Accordingly, $\mathsf{F}(\sigma, \cdot)$ vanishes at $\beta = 0$, is monotonically increasing in the interval $(0, \alpha^*(\sigma))$, then is monotonically decreasing in the interval $(\alpha^*(\sigma), 1)$ and tends to $-\infty$ when β goes to 1. Hence there exists a unique value of β in $(0, 1)$, say $\bar{\alpha}(\sigma)$, for which F vanishes:

$$\mathsf{F}(\sigma, \bar{\alpha}(\sigma)) = 0, \quad \alpha^*(\sigma) < \bar{\alpha}(\sigma) < 1. \tag{125}$$

Since $\alpha > 0$ and $\ell\alpha' = \pm\sqrt{\mathsf{F}(\sigma,\alpha)}$ in $\mathcal{S}_i$, by standard arguments for this type of ordinary differential equations, one deduces that $\alpha'(x_i) = 0$, $\alpha(x_i) = \bar{\alpha}(\sigma)$ and

$$\ell\alpha' = \begin{cases} +\sqrt{\mathsf{F}(\sigma,\alpha)} & \text{in } (x_i - D, x_i) \\ -\sqrt{\mathsf{F}(\sigma,\alpha)} & \text{in } (x_i, x_i + D). \end{cases} \tag{126}$$

In other words, $\bar{\alpha}(\sigma)$ corresponds to the maximal value of damage (at the given time), taken at the center of the localization zone. The damage state $\alpha^*(\sigma)$ corresponds to the damage state of the bar under the same stress during a homogeneous damage process, see (115). This means that the center part of any localization damage zone is more damaged while the remaining part of the bar is less damaged than in a homogeneous process at the same stress level.

The size of the localization zone is deduced from (126) and (121) by integration. It also depends only on σ and is given by

$$D(\sigma) = \ell \int_0^{\bar{\alpha}(\sigma)} \frac{d\beta}{\sqrt{2w(\beta)/w_1 - \sigma^2\left(S(\beta) - S_0\right)/w_1}}. \tag{127}$$

Hence $D(\sigma)$ is proportional to the internal length and is finite because the integral is convergent [3]. Provided that $L \geq 2D(\sigma)$, it is thus possible to insert a localization zone of size $2D(\sigma)$ inside the bar. The position x_i of the center can be chosen arbitrarily in the interval $[D(\sigma), L - D(\sigma)]$.

We finally deduce from (126) and (121) that, in the localization zone, the damage field is given by the following implicit relation between x and α:

$$|x - x_i| = \ell \int_\alpha^{\bar{\alpha}(\sigma)} \frac{d\beta}{\sqrt{2w(\beta)/w_1 - \sigma^2\left(S(\beta) - S_0\right)/w_1}}. \tag{128}$$

The damage field is symmetric with respect to the center of the localization zone, decreasing continuously from $\bar{\alpha}(\sigma)$ at the center to 0 at the boundary. The spatial regularity of the damage profile is governed by the regularity of the constitutive functions $\alpha \mapsto w(\alpha)$ and $\alpha \mapsto S(\alpha)$. Under Hypothesis 1, $x \mapsto \alpha(x)$ as a solution of (120) is at least three times continuously differentiable in $\mathcal{S}_i$ provided that $\sigma \in (0, \sigma_{\mathsf{c}})$. The damage profile is even indefinitely differentiable when the constitutive functions are. We will see in the next subsection that this regularity is lost at the limit $\sigma = 0$.

Remark 10. *The size of an inner localization zone and the damage localization profile depend only on σ. Since σ is a global quantity, all the inner localization zones have the same size and the same profile at a given time. One can also consider localization zones which start at the boundary. In such a case, the consistency condition at the boundary enforces that $\alpha_t'\dot{\alpha}_t$ vanishes at the boundary and consequently the profile is still given by* (127)-(128) *with $x_i = 0$ or $x_i = L$ and $x \in [0, D(\sigma)]$ or $x \in [L - D(\sigma), L]$. In other words, the profile of a localization zone starting*

[3] Indeed, $\mathsf{F}(\sigma,\beta)$ behaves like $\frac{\partial \mathsf{F}}{\partial \beta}(\sigma,0)\beta$ near $\beta = 0$ and like $\frac{\partial \mathsf{F}}{\partial \beta}(\sigma,\bar{\alpha}(\sigma))(\beta - \bar{\alpha}(\sigma))$ near $\beta = \bar{\alpha}(\sigma)$. Since $\frac{\partial \mathsf{F}}{\partial \beta}(\sigma,0) > 0$ and $\frac{\partial \mathsf{F}}{\partial \beta}(\sigma,\bar{\alpha}(\sigma)) < 0$, the integral is convergent.

at the boundary is a half of the profile of an inner localization zone, see Figure 7. Accordingly, the total length of the set $\mathcal{S}$ of localization zones is $nD(\sigma)$ with n the number of half-*localization zones. Note, however, that such a half-localization zone becomes impossible when one changes the boundary conditions and does not allow that the end of the bar be damaged.*

We can summarize our construction of a localized solution by the following property:

Property 11.1 (Profile of a localized damage field)**.** *For a given stress $\sigma \in (0, \sigma_c)$, the damage field in an inner localized damage zone $(x_i - D(\sigma), x_i + D(\sigma))$ is given by* (128) *while the half-length $D(\sigma)$ of the localized damage zone is finite, proportional to the internal length ℓ and given by* (127)*. The damage profile is symmetric with respect to the center x_i of the localized damage zone, maximal at the center, the maximal value $\bar{\alpha}(\sigma)$ being given by* (125)*. The damage profile is a continuously differentiable function of x, decreasing from $\bar{\alpha}(\sigma)$ at the center to 0 at the boundary of the localized damage zone. The matching with the undamaged part of the bar is smooth, the damage and the gradient of damage vanishing at the boundary of the localized damage zone, see Figure 7.*

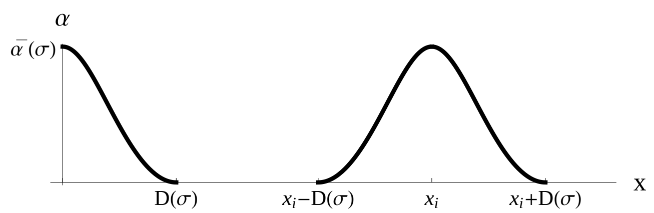

Figure 7. A typical damage profile in an inner localization zone and in a boundary localization zone when $0 < \sigma < \sigma_c$

11.2. Dependency of the damage profile on the overall stress

The maximal value of damage depends only on σ and enjoys the following property:

Property 11.2 (Variation of the maximal value of the damage with the stress)**.** *When σ decreases from σ_c to 0, the maximal value $\bar{\alpha}(\sigma)$ taken by the damage field at the center of a localization zone increases from 0 to 1.*

Proof. Let $0 < \sigma_1 < \sigma_2 < \sigma_c$. Since $0 = \mathsf{F}(\sigma_1, \bar{\alpha}(\sigma_1)) = \mathsf{F}(\sigma_2, \bar{\alpha}(\sigma_2)) < \mathsf{F}(\sigma_1, \bar{\alpha}(\sigma_2))$, and since $\mathsf{F}(\sigma_1, \beta) < 0$ when $\bar{\alpha}(\sigma_1) < \beta < 1$, we have $\bar{\alpha}(\sigma_1) > \bar{\alpha}(\sigma_2)$. Hence $\sigma \mapsto \bar{\alpha}(\sigma)$ is decreasing.

Let us prove that $\bar{\alpha}_0 := \lim_{\sigma \to \sigma_c} \bar{\alpha}(\sigma) = 0$. The limit exists and is non-negative because $\sigma \mapsto \bar{\alpha}(\sigma)$ is monotone and positive on $(0, \sigma_c)$. Passing to the limit in $\mathsf{F}(\sigma, \bar{\alpha}(\sigma)) = 0$ when σ goes to 0 gives $\mathsf{F}(\sigma_c, \bar{\alpha}_0) = 0$. Since $\mathsf{F}(\sigma_c, 0) = 0$ and $\partial\mathsf{F}/\partial\beta(\sigma_c, \beta) < 0$ for $\beta > 0$, we can conclude that $\bar{\alpha}_0 = 0$.

Let us prove that $\bar{\alpha}_1 := \lim_{\sigma\to 0}\bar{\alpha}(\sigma) = 1$. The limit exists and belongs to $(0,1]$ because $\bar{\alpha}(\sigma)$ is decreasing and belongs to $(0,1)$. If $\bar{\alpha}_1 < 1$, passing to the limit in $\mathsf{F}(\sigma,\bar{\alpha}(\sigma)) = 0$ when σ goes to 0 gives $0 = \mathsf{F}(0,\bar{\alpha}_1) = 2w(\bar{\alpha}_1)$, a contradiction. Hence $\bar{\alpha}_1 = 1$. □

Let us consider the limiting cases $\sigma = \sigma_\mathsf{c}$ or $\sigma = 0$, *i.e.* the onset of the damage localization and the moment at which the bar breaks.

Case $\sigma = \sigma_\mathsf{c}$. In such a case, the differential system (120)-(121) governing the damage profile in a damage localization zone becomes

$$-{\sigma_\mathsf{c}}^2 S'(\alpha) + 2w'(\alpha) - 2w_1\ell^2\alpha'' = 0 \quad \text{in} \quad \mathcal{S}_i, \qquad \alpha = \alpha' = 0 \quad \text{on} \quad \partial\mathcal{S}_i.$$

Integrating the differential equation over $\mathcal{S}_i$ and using the boundary conditions lead to

$${\sigma_\mathsf{c}}^2 \int_{\mathcal{S}_i} S'(\alpha)dx = 2\int_{\mathcal{S}_i} w'(\alpha)dx.$$

But, by Hypothesis 1, since ${\sigma_\mathsf{c}}^2 S'(\alpha) \leq 2w'(\alpha)$ for all $\alpha \in [0,1]$ and since the equality holds if and only if $\alpha = 0$, the unique solution of the differential equation is $\alpha(x) = 0$ for all x. This is in agreement with the previous analysis where it was shown that the amplitude of the damage profile tends to 0 when σ goes to σ_c. It means that the onset of the damage localization process is progressive as a function of the overall stress. To find the shape of the damage profile when σ is close to σ_c, one way is to expand the solution (127)-(128) in terms of the small parameter ${\sigma_\mathsf{c}}^2 - \sigma^2$. This requires a careful analysis of the behavior of the integrals when σ goes to σ_c. One can show for instance that $\lim_{\sigma\uparrow\sigma_\mathsf{c}} D(\sigma) = D_c > 0$ whereas $\lim_{\sigma\uparrow\sigma_\mathsf{c}}\bar{\alpha}(\sigma) = 0$. This means that the onset of damage localization is of small amplitude but occurs in a zone of *finite* length. This result can be obtained directly by considering the *bifurcation* equation. Let us follow this latter way as it is customary in bifurcation problems.

Specifically, since we are seeking for small damage fields for σ close to σ_c, one linearizes the differential equation (120) which becomes

$$\alpha > 0 \qquad \text{and} \qquad ({\sigma_\mathsf{c}}^2 S''(0) - 2w''(0))\alpha + 2w_1\ell^2\alpha'' = ({\sigma_\mathsf{c}}^2 - \sigma^2)S'(0) \quad \text{in} \quad \mathcal{S}_i = (x_i - D_c, x_i + D_c) \tag{129}$$

the boundary conditions remaining unchanged. To obtain (129) we have taken into account that $S'(0){\sigma_\mathsf{c}}^2 = 2w'(0)$. This is the desired *bifurcation* equation. Then by standard arguments and after easy calculations which are left to the reader, one gets

$$\alpha(x) = \frac{2S'(0)({\sigma_\mathsf{c}}^2 - \sigma^2)}{{\sigma_\mathsf{c}}^2 S''(0) - 2w''(0)}\cos^2\frac{\pi(x - x_i)}{2D_c}, \qquad D_c = \pi\ell\sqrt{\frac{2w_1}{{\sigma_\mathsf{c}}^2 S''(0) - 2w''(0)}}. \tag{130}$$

One has thus obtained the following property:

Property 11.3 (The onset of a damage localization process)**.** *A localization of damage can occur when the stress has reached the critical value σ_c given by* (116)*. Damage then appears in* *one (or several) zones of finite size whose half-length* D_c *is given by* (130)*, with a profile which is approximately a sinusoid whose amplitude progressively increases when the stress decreases, see* (130)*.*

Note that this value of D_c is the same as in (118), which simply means that D_c is the minimal length of the bar for which one can construct a non-homogeneous solution at the end of the elastic phase. The localized solution which requires less space is of course the one which starts at one boundary, its size being half of the size of an inner localization zone, see Remark 10.

Example 6. *In the case of the family of models of Example 5, the half-length of the damage zone and the amplitude of the damage profile at the onset of damage are given by*

$$\bar{\alpha}(\sigma) = \frac{2}{p+q}\left(1 - \frac{\sigma^2}{\sigma_c{}^2}\right), \qquad D_c = \sqrt{\frac{2}{(p+q)q}\frac{\pi\ell}{\varepsilon_c}}.$$

Case $\sigma = 0$**.** In this case, our previous construction of the damage profile is not valid. Indeed, the differential system (120)-(121) becomes

$$\alpha > 0 \quad \text{and} \quad w'(\alpha) - E_0\ell^2\alpha'' = 0 \quad \text{in} \quad \mathcal{S}_i, \qquad \alpha = \alpha' = 0 \quad \text{on} \quad \partial\mathcal{S}_i.$$

Integrating the differential equation over $\mathcal{S}_i$ and using the boundary conditions leads to $\int_{\mathcal{S}_i} w'(\alpha)dx = 0$, which is impossible by Hypothesis 1. As suggested by the fact that the maximal value of damage tends to 1 when σ goes to 0, one has to search for profile such that the damage field takes the value 1 at the center of the zone. Since some quantities like the compliance function $\alpha \mapsto S(\alpha)$ and its derivatives become infinite when α goes to 1, the regularity of the damage field is lost and $\alpha'(x)$ is no more defined at $x = x_i$ but undergoes a jump discontinuity. So the differential system now reads

$$\alpha > 0 \quad \text{and} \quad w'(\alpha) - E_0\ell^2\alpha'' = 0 \quad \text{in} \quad \mathcal{S}_i \setminus x_i, \qquad \alpha(x_i) = 1, \qquad \alpha = \alpha' = 0 \quad \text{on} \quad \partial\mathcal{S}_i.$$

Multiplying by α' the differential equation valid on each half-zone and taking into account the boundary conditions at the ends, one still obtains a first integral $w_1\ell^2\alpha'(x)^2 = 2w(\alpha(x))$ in $\mathcal{S}_i \setminus x_i$. Since $\alpha > 0$ in $\mathcal{S}_i$, denoting by D_0 the half-length of the localization zone, one necessarily has

$$\ell\alpha' = \begin{cases} +\sqrt{2w(\alpha)/w_1} & \text{in } (x_i - D_0, x_i) \\ -\sqrt{2w(\alpha)/w_1} & \text{in } (x_i, x_i + D_0). \end{cases}$$

Since $\alpha(x_i) = 1$, the jump of α' at x_i is equal to $-2\sqrt{2S_0 w(1)}/\ell$. By integration, we obtain the damage profile and the half-length of the localization zone:

$$|x - x_i| = \ell\int_\alpha^1 \frac{d\beta}{\sqrt{2w(\beta)/w_1}}, \qquad D_0 = \ell\int_0^1 \frac{d\alpha}{\sqrt{2w(\alpha)/w_1}}. \tag{131}$$

One can remark that this solution can be obtained formally by taking $\sigma = 0$ and $\bar{\alpha}(0) = 1$ in (127)-(128). We have proved the following

Property 11.4 (Rupture of the bar at the center of a localization zone)**.** *At the end of the damage process, when the stress has decreased to 0, the damage takes the critical value* 1 *at the center of the localized damage zone. The damage profile and the half length* D_0 *of the damage zone are then given by* (131)*. The profile is still symmetric and continuously decreasing to 0 from the center to the boundary, but its slope is discontinuous at the center.*

Example 7. *In the cases of the family of models of Example 5, the half-length of the damage zone and the amplitude of the damage profile when the bar breaks are given by*

$$|x - x_i| = \frac{\ell}{\varepsilon_c}\sqrt{\frac{p}{q}}\int_0^{1-\alpha}\frac{dv}{\sqrt{1-v^p}}, \qquad D_0 = \frac{\ell}{\varepsilon_c}\sqrt{\frac{p}{q}}\int_0^{1}\frac{dv}{\sqrt{1-v^p}}.$$

For $p = 1$*, the profile is made of two symmetric arcs of parabola:*

$$\alpha(x) = \left(1 - \frac{|x - x_i|}{D_0}\right)^2, \qquad D_0 = \frac{2\ell}{\varepsilon_c\sqrt{q}}.$$

For $p = 2$*, the profile is made of two symmetric arcs of sinusoid:*

$$\alpha(x) = 1 - \sin\frac{\pi|x - x_i|}{2D_0}, \qquad D_0 = \frac{\pi\ell}{\varepsilon_c\sqrt{2q}}.$$

The greater p is, the greater the size of the damage zone and the damage field, see Figure 8.

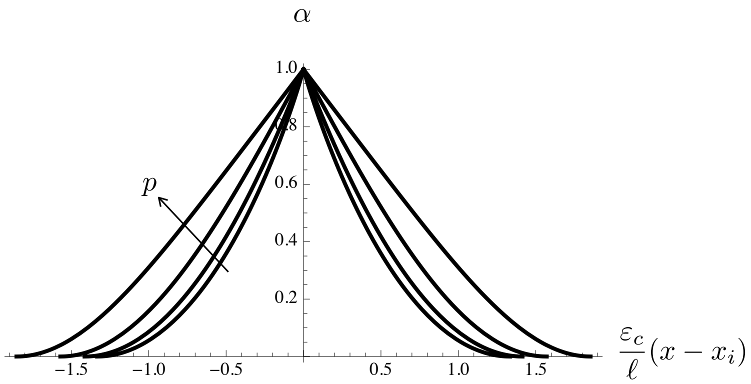

Figure 8. Damage profile in the localization zone when the bar breaks, for $q = 4$ and different values of the parameter p ($p = 1/2, 1, 2, 4$) in the family of brittle materials of Example 5

As we will see in the next subsection, the growth of the localization damage zone when the stress decreases is essential in order to satisfy the irreversibility condition. However, if we compare the size of the damage zone at the onset of damage with its size when the bar breaks, *i.e.* if we compare D_c with D_0, it is not clear whether $D_0 \geq D_c$. If we consider, for instance, the family of models of Example 5 one has

$$\frac{D_0}{D_c} = \frac{\sqrt{p(p+q)}I_p}{\pi\sqrt{2}}, \qquad I_p = \int_0^1 \frac{dv}{\sqrt{1-v^p}}.$$

Since $q \geq p > 0$, the inequality $D_0 \geq D_c$ holds only if $pI_p \geq \pi$. But, since pI_p is an increasing function of p which is equal to π when $p = 2$, one has $D_0 \geq D_c$ if and only if $p \geq 2$. If we plot the graph of $\sigma \mapsto D(\sigma)$ for different values of the parameters, one sees that $D(\sigma)$ is a decreasing function of σ only for large enough values of the parameters p and q, see Figure 9. So, the

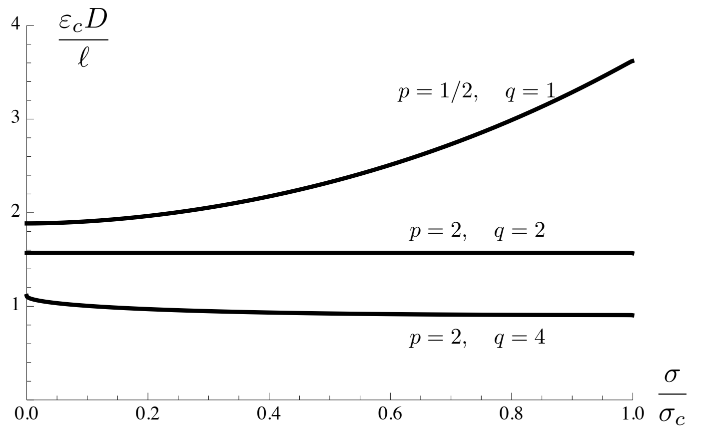

Figure 9. Evolution of the half-length of the damage zone as a function of the stress for three models of the family of brittle materials of Example 5: for $p = 1/2$ and $q = 2$, D decreases all along the damage process; for $p = q = 2$, D remains constant; for $p = 2$ and $q = 4$, D increases all along the damage process.

monotonicity of $\sigma \mapsto D(\sigma)$ is a material property which depends only on the two state functions $\alpha \mapsto E(\alpha)$ and $\alpha \mapsto w(\alpha)$. Note that the value of the internal length does not play a role. Since the study of the general case is quite difficult, we will merely establish a sufficient condition for the monotonicity of $\sigma \mapsto D(\sigma)$ in the case of perfectly brittle materials.

Property 11.5 (Variation of the size of a localization zone with the stress.)**.** *For perfectly brittle materials in the sense of Hypothesis 2, if $\alpha \mapsto \sqrt{E(\alpha)}$ is convex, then $\sigma \mapsto D(\sigma)$ is non increasing. In particular, this condition is satisfied in the family of models of Example 5 when $p = q \geq 2$.*

Proof. Let us consider a perfectly brittle material and set $\sigma_\mathsf{c} = E_0\varepsilon_c$, $S_0 = 1/E_0$. For $\sigma \in (0, \sigma_\mathsf{c})$, the function F defined in (124) now reads

$$\mathsf{F}(\sigma, \alpha) = S_0 \left(S_0 {\sigma_\mathsf{c}}^2 E(\alpha) - \sigma^2\right) \frac{1 - S_0 E(\alpha)}{E(\alpha)}, \qquad \alpha \in [0, 1).$$

The maximal value $\bar{\alpha}(\sigma)$ of the damage is hence given by

$$\bar{\alpha}(\sigma) = E^{-1}\left(\frac{\sigma^2}{{\sigma_\mathsf{c}}^2} E_0\right).$$

Inserting into the definition of $D(\sigma)$ yields

$$D(\sigma) = \ell \int_0^{\bar{\alpha}(\sigma)} \sqrt{\frac{E_0 E(\alpha)}{\left(S_0 {\sigma_\mathsf{c}}^2 E(\alpha) - \sigma^2\right)\left(1 - S_0 E(\alpha)\right)}} d\alpha.$$

Let us consider the change of variable $\alpha \mapsto \theta$ at given σ:

$$S_0 E(\alpha) = 1 - \theta + \theta \frac{\sigma^2}{{\sigma_\text{c}}^2}.$$

θ increases from 0 to 1 when α increases from 0 to $\bar{\alpha}(\sigma)$. Making this change of variable in the integral giving $D(\sigma)$ yields

$$D(\sigma) = \frac{\ell}{2\varepsilon_c} \int_0^1 \frac{d\theta}{|\Phi'(\alpha)|\sqrt{\theta(1-\theta)}},$$

where Φ stands for the function $\alpha \mapsto \Phi(\alpha) := \sqrt{S_0 E(\alpha)}$ and $\Phi'(\alpha)$ is the derivative of Φ at α. If $\alpha \mapsto \sqrt{E(\alpha)}$ is convex, then Φ' is a non-decreasing function of α. Since E is a decreasing function of α, so is Φ and $|\Phi'(\alpha)| = -\Phi'(\alpha)$. Hence $|\Phi'(\alpha)|$ is a non increasing function of α. Since α is a decreasing function of σ at given $\theta \in (0,1)$, $|\Phi'(\alpha)|$ is a non-decreasing function of σ at given θ. Accordingly D is a non-increasing function of σ.

In the case of Example 5, the material is perfectly brittle when $p = q > 0$ and then $E(\alpha) = E_0(1-\alpha)^p$. Hence, $\alpha \mapsto \sqrt{E(\alpha)}$ is convex when $p \geq 2$. Note that when $p = 2$, a straightforward calculation gives

$$D(\sigma) = \frac{\pi\ell}{2\varepsilon_c},$$

and hence the size of the localization zone remains fixed all along the damage process. □

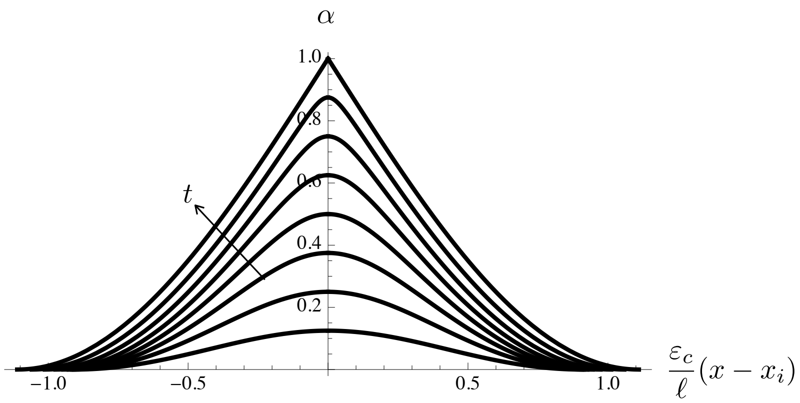

Figure 10. The damage profile for a given t and its evolution with t by assuming that $t \mapsto \sigma_t$ is decreasing in the case of the model of Example 5 with $p = 2$ and $q = 4$. The rupture occurs when $\sigma_t = 0$ and $\bar{\alpha}(\sigma_t) = 1$. We check numerically that $\sigma \mapsto D(\sigma)$ is decreasing, see Figure 9.

11.3. Checking the irreversibility

It remains to check that the localized damage fields that we have constructed at different values of σ lead to an evolution in time which satisfies the irreversibility condition $\dot{\alpha} \geq 0$. Let us reintroduce the time and the index t in the notation. Since the center of the localization zone is fixed, the condition of irreversibility is satisfied only if $t \mapsto \bar{\alpha}(\sigma_t) = \alpha_t(x_i)$ is non-decreasing. Since $\sigma \mapsto \bar{\alpha}(\sigma)$ is decreasing, this is possible only if $t \mapsto \sigma_t$ is non-increasing. Since $\alpha_t(x_i, x_i + D(\sigma_t)) = 0$ and since $\alpha_t(x) > 0$ for $|x - x_i| < D(\sigma_t)$ by construction, the condition of irreversibility is satisfied only if $t \mapsto D(\sigma_t)$ is non decreasing. This requires that $\sigma \mapsto D(\sigma)$ is non increasing, condition which is not automatically satisfied by the damage model, see Figure 9. When this condition is not satisfied, our construction of localized solutions is no more valid. We must consider an evolution of the damage where a part of the localization zone reenters in a non damaging phase, the size of the still damaging part decreasing with time. To avoid such a situation we make the following hypothesis:

Hypothesis 3. *We assume that $\alpha \mapsto E(\alpha)$ and $\alpha \mapsto w(\alpha)$ are such that $\sigma \mapsto D(\sigma)$ is non increasing.*

Note that this hypothesis is satisfied in the class of models of Example 5 when $p = q \geq 2$ by virtue of Property 11.5. Under this condition, it is possible to obtain the following property:

Property 11.6. *Under Hypothesis 3, in order that $t \mapsto \alpha_t$ given by (128) in a localization zone (and equals to 0 otherwise) is non-decreasing, it is necessary and sufficient that $t \mapsto \sigma_t$ is non-increasing.*

Proof. We know that it is necessary, it remains to prove that it is sufficient. Let us assume that $t \mapsto \sigma_t$ is non increasing. Then $t \mapsto \bar{\alpha}(\sigma_t)$ and $t \mapsto D(\sigma_t)$ are non decreasing. Let $t_1 < t_2$ and x be such that $|x - x_i| \leq D(\sigma_{t_1})$. It is sufficient to prove that $\alpha_2 := \alpha_{t_2}(x) \geq \alpha_{t_1}(x) =: \alpha_1$. Owing to (128), since F is a decreasing function of σ and since $\sigma_{t_2} \leq \sigma_{t_1}$, we have

$$0 \leq D(\sigma_{t_2}) - D(\sigma_{t_1}) = \int_0^{\alpha_2} \frac{\ell\, d\beta}{\sqrt{\mathsf{F}(\sigma_{t_2}, \beta)}} - \int_0^{\alpha_1} \frac{\ell\, d\beta}{\sqrt{\mathsf{F}(\sigma_{t_1}, \beta)}} \leq \int_{\alpha_1}^{\alpha_2} \frac{\ell\, d\beta}{\sqrt{\mathsf{F}(\sigma_{t_1}, \beta)}}.$$

Hence $\alpha_2 \geq \alpha_1$. □

By virtue of this last property, our construction of a non homogeneous solution is valid provided that the bar is sufficiently long for a localization zone to appear and grow without reaching the boundary. Since the size of the localization zone increases with t, that leads to the inequality $L \geq 2D_0$. If we consider a non homogeneous solution which starts at one end, our construction is valid provided that the localization zone does not reach the other end of the bar and hence provided that $L \geq D_0$. Owing to (131), that gives the following lower bound for L:

$$L \geq \ell \int_0^1 \frac{d\alpha}{\sqrt{2w(\alpha)/w_1}}. \tag{132}$$

Proposition 11.7 (A solution of the evolution problem with damage localization). *Under Hypotheses 1 and 3, we have constructed a damage evolution $t \mapsto \alpha_t$ which satisfies the evolution problem* **PB 10.1** *if the bar is long enough and if we can control the loading in such a manner that the stress is continuously decreasing. A typical example of the evolution of the damage from its onset to the rupture is given in Figure 10.*

In the case where Hypothesis 3 is not satisfied, our construction is no more valid, the irreversibility condition is not satisfied because the size of the damage zone is decreasing. In such a case, the construction of the solution must be refined. It is outside the scope of these Lectures.

11.4. Energy dissipated in a localization zone

By virtue of Property 10.2 and of (110) the energy dissipated in an inner localization zone when the stress is σ is given by

$$\mathcal{E}_d(\sigma) = \int_{x_i - D(\sigma)}^{x_i + D(\sigma)} \left(\frac{1}{2} w_1 \ell^2 \alpha'(x)^2 + w(\alpha(x)) \right) dx.$$

By symmetry, it is twice the energy dissipated in a half-zone. Using the change of variable $x \to \alpha$ and (123), we obtain

$$\mathcal{E}_d(\sigma) = \ell \int_0^{\bar{\alpha}(\sigma)} \frac{4w(\alpha) - \sigma^2(S(\alpha) - S_0)}{\sqrt{2w(\alpha)/w_1 - \sigma^2(S(\alpha) - S_0)/w_1}} d\alpha. \tag{133}$$

It is easy to check that $\sigma \mapsto \mathcal{E}_d(\sigma)$ is decreasing with $\mathcal{E}_d(\sigma_c) = 0$, while $\mathcal{E}_d(0)$ represents the energy dissipated in a localization zone during the process of damage up to rupture. Let us call *fracture energy* and denote by G_c this energy by reference to the Griffith surface energy density in Griffith's theory of fracture. Since $\bar{\alpha}(0) = 1$, we have

Property 11.8 (Fracture energy). *The energy dissipated in an inner localization zone during the damage process up to rupture is a material constant G_c which is given by*

$$G_c = \ell \int_0^1 \sqrt{8w(\alpha)/w_1} d\alpha. \tag{134}$$

Because of the lack of constraint on the damage at the boundary, the dissipated energy in a boundary localization zone up to the rupture is $G_c/2$.

Example 8. *In the case of the family of strongly brittle materials of Example 5, the fracture energy is given by*

$$G_c = 2J_p \sqrt{\frac{q}{p}} \sigma_c \ell, \qquad J_p = \int_0^1 \sqrt{1 - v^p}\, dv. \tag{135}$$

Thus G_c is proportional to the product of the critical stress by the internal length, the coefficient of proportionality depending on the exponents p and q. This link between surface fracture energy, *critical stress and internal length is quite similar to the link between the analogous quantities in cohesive force models, see (*Charlotte et al., 2000*;* Marigo and Truskinovsky, 2004*). More generally, cohesive force models and gradient damage models have very similar properties. Both can be seen as regularization of Griffith's model in fracture mechanics, with in each case the great advantage (by comparison to Griffith's model)of containing a critical stress and an internal length. A more fundamental comparison of these two regularized models deserves to be made. The interested reader can refer to (*Lorentz et al., 2011*) for a first interesting attempt in this direction.*

12. Concluding remarks

The solution of the evolution problem is characterized by the competition between two fundamental damage modes: solutions homogeneous in space and solutions with damage localization. The study of their properties is fundamental to have a qualitative understanding of an evolution problem. We focused on a class of damage models characterized by a finite elastic limit and stress-softening. In this case, for sufficiently long bars, bifurcation from the homogeneous solution toward a localized appears as soon as the stress reaches the elastic limit

$$\sigma_{\mathsf{c}} = \sqrt{\frac{2w'(0)}{S'(0)}}. \tag{136}$$

We studied the evolution of the bifurcated branch until the complete failure of the bar, which is obtained when the maximum value of the damage field reaches 1. The fully localized solution is characterized by a localization width D_0 and a dissipated energy G_c given by:

$$D_0 = c_{1/w}\,\ell,, \qquad G_c = c_w\, w_1\,\ell. \tag{137}$$

where $c_{1/w}$ and c_w are two dimensionless constants defined by

$$c_{1/w} := \sqrt{2}\int_0^1 \sqrt{\frac{w_1}{w(\alpha)}}d\alpha, \qquad c_w = 2\sqrt{2}\int_0^1 \sqrt{\frac{w(\alpha)}{w_1}}d\alpha \tag{138}$$

The original expression of the energy density (84) was written taking as reference value for the dissipation the energy dissipated per unit length in a uniform solution, w_1. On the contrary the characteristic dissipation per unit length in a localized solution is $G_c = c_w\, w_1\,\ell/L$, where L is the length of the bar. Evidently, for small values of ℓ/L, damage localization is a convenient failure mode. Indeed solutions with homogeneous non null damage are unstable for long bars. If one want to focus on the failure induced by localized solutions, it is convenient to use equation (137) to rewrite the energy density in the following form, where G_c appears as independent variable instead of w_1

$$\mathsf{W}(\varepsilon,\alpha,\alpha') = \frac{1}{2}E(\alpha)\varepsilon^2 + \frac{G_c}{c_w}\left(\frac{w(\alpha)}{w_1\ell} + \frac{\ell\,\alpha'^2}{2}\right). \tag{139}$$

This form of the energy density underlines the links between the damage and brittle fracture.

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